N-tuple Classifier

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Solving Complex Computational Problems

- Break global problem into smaller subproblems
- Each of which can be solved independently
- Optimally solve the subproblems
- Combine the solutions to the subproblems to obtain the solution to the global problem

Decompositions

- Maximize the Dependencies within each of the smaller problems
- Maximize the Independence between each of the smaller problems

Decompositions

- Recursive Decomposition
- Data Decomposition
- Functional Decompositions
- Search Space Decompositions

Decompositions and Optimality

- Sometimes the Solution to the decomposed problem is optimal
- Sometimes the Solution to the decomposed problem is sub-optimal
- The Solution obtained by decomposition can be close to optimal

The Subspace Classifier

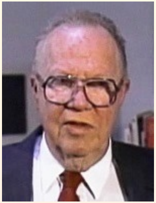
Definition

A Subspace Classifier is one that projects the measurement vector to one or more subspaces where the projected vector is processed and then the processed projected vectors are combined in a way to form an assigned classification.

It is typical for the projection operators to be orthogonal projection operators. It is not unusual for the projection operators to be axis aligned.

N-Tuple Method - Bledsoe and Browning -

(a) Bledsoe

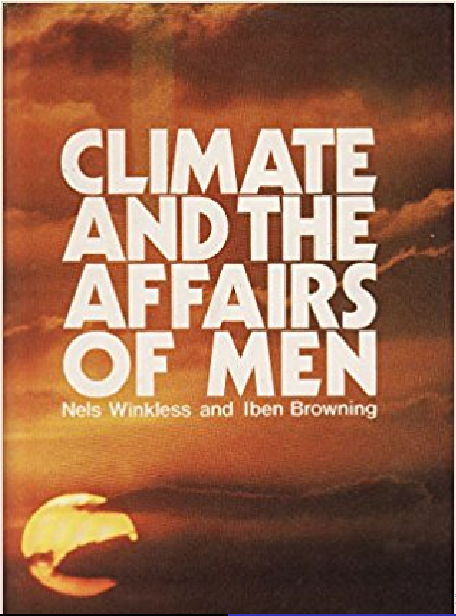


(b) Browning

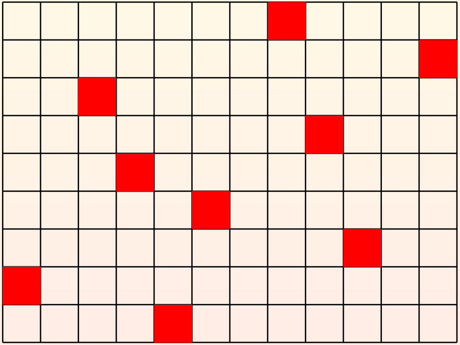
- Developed For Printed Character Recognition
- Each character is contained in an image of $I \times J$ pixels
- Each pixel is a binary 1 or a binary 0
- Designed for table lookup hardware

W.W. Bledsoe and I. Browning, *Pattern Recognition and Reading by Machine*, **Proceeding Eastern Joint Computer Conference**, Boston, 1959, 232-255.

Climate and the Affairs of Men



N-Tuple Method



Bledsoe and Browning had an array of 10×15 pixels
In general, N Randomly Chosen Pixel Positions

N-Tuple Method

- A small number of pixel positions are randomly selected for each subspace
  - Bledsoe and Browning selected 2 pixels at a time
- Have multiple sets of such randomly selected pixel positions
  - Bledsoe and Browning selected 75 sets of randomly selected mutually exclusive pixel pairs
  - Each subspace was two dimensions, there were 4 possible values in each subspace dimension
- Each of the pixel positions had been thresholded (quantized) and contained a binary 0 or a binary 1

N-Tuple Method

- Concatenate all the binary values to form a binary number as an address for the subspace
- The memory required for each subspace for each class was 2 dimensions $\times$ 4 possible values per subspace
  - Number of classes: 26 letters, 10 digits make up 36 classes
  - For each of 36 classes
  - Bledsoe and Browning implementation needed 8 locations for each of the 75 two dimensional subspaces for each class
  - The number of memory locations was 600 for each of 36 classes
- Use the 4 bit numbers to access an address in memory
- For each subspace
- For each character class

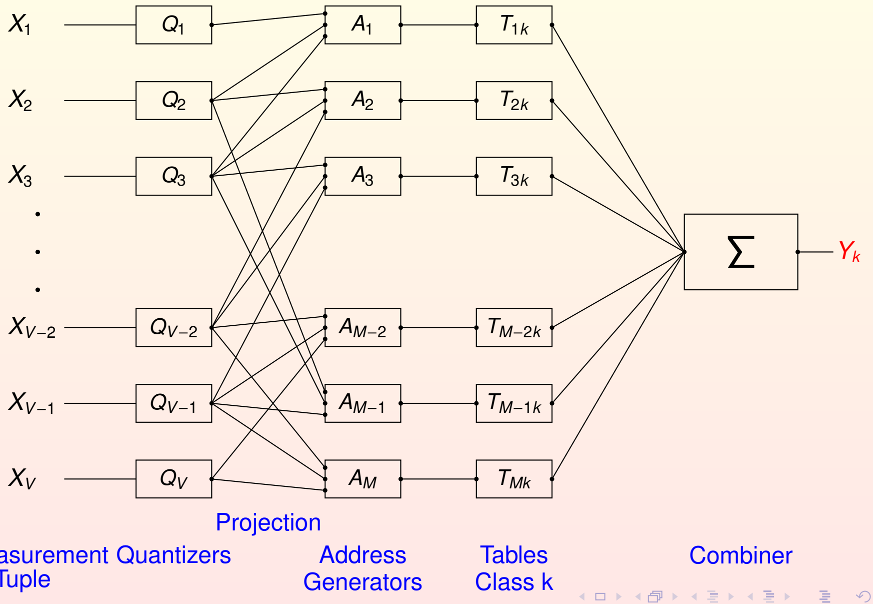
N-Tuple Method

- M pattern sets of N randomly selected pixel positions
- A printed character produces M N-digit binary numbers $b_1, \ldots, b_M$
- K character classes
- T_{mk} lookup table for pattern set m and class k
- $T_{mk}(b_m)$ holds the fraction of times a character in the training set of class k has the binary number b_m for the m^{th} pattern set
- Compute
  - $f_k = \prod_{m=1}^{M} T_{mk}(b_m)$
  - $f_k = \sum_{m=1}^{M} T_{mk}(b_m)$
- Assign the character to unique class k, if there is one, for which $f_k > 0$ is highest
- Otherwise reserve decision

An Alternate N-Tuple Method

- M pattern sets of N randomly selected pixel positions
- A printed character produces M binary numbers $b_1, \ldots, b_M$
- K character classes
- T_m lookup table for pattern set m
- $T_m(b_m)$ holds the subset of classes most associated with the binary number b_m for the m^{th} pattern set
- Compute
  - $f = \cap_{m=1}^{M} T_m(b_m)$
  - Assign the character to unique class k, if there is one, where $k \in f$ and $|f| = 1$
  - Otherwise reserve decision

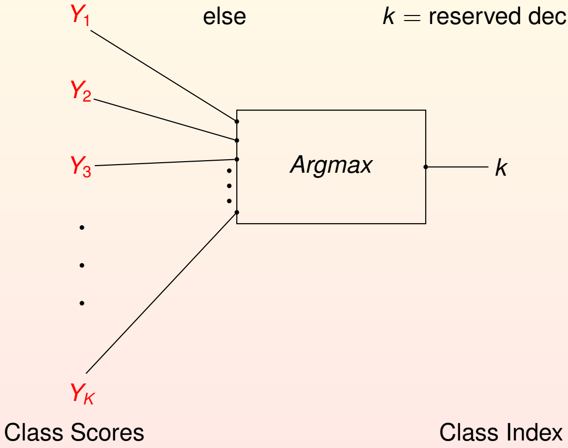
The N-tuple Calculation for Class k



The N-tuple Class Index Generator

if $Y_k > Y_i,\ i \neq k,\ k = Argmax\{Y_1, Y_2, \ldots, Y_K\}$

else $k = \text{reserved decision}$



The Need For The Indexed Tuple

Consider the five dimensional measurement vector (a, b, c, d, e) where

- a is the value produced by feature f_1
- b is the value produced by feature f_2
- $\vdots$
- e is the value produced by feature f_5

The Need For the Indexed Tuple

- Project the measurement vector (a, b, c, d, e) to the third and fifth feature
- The resulting tuple is (c, e).
- But now we have lost from which features c and e came.

In the database world, every value comes from a field and the connection between field and value is never lost.

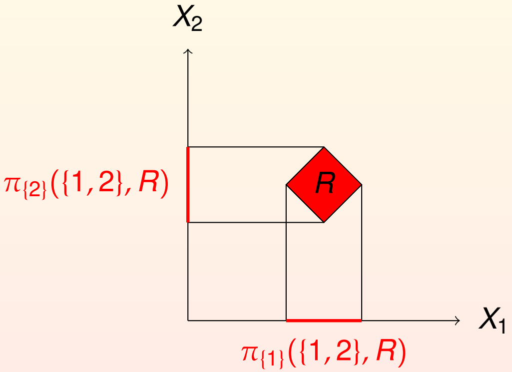
The Indexed Tuple

- Index Sets serve as Field Names
- The tuple (a, b, c, d, e) is written as $(\{1, 2, 3, 4, 5\}, (a, b, c, d, e))$
- (c, d) is written as $(\{3, 4\}, (c, d))$
- (a, b, e) is written as $(\{1, 2, 5\}, (a, b, e))$
- A tuple list $R = \langle (a, b, e), (q, r, s), (t, x, z) \rangle$ is written as $(\{1, 2, 5\}, R)$
  - First component is an index set for the features
  - Second component is a set of tuples
  - Each component of a tuple is the value for the corresponding indexed features

The Tuple Projection Operator

- Suppose that S is a tuple list with respect to the index set I
  - (I, S)
- Let $J \subset I$.
- The projection of (I, S) from the space indexed by I to the subspace indexed by J
  - $\pi_J(I, S) = (J, R)$

Projection



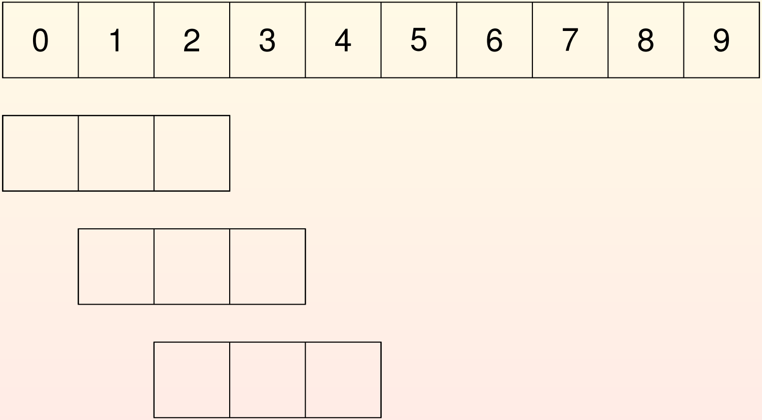
N-tuple Method Using Index Sets and Projections

- Index Set $I = \{1, \ldots, V\}$
- $X_1, \ldots, X_V$ are the V quantized features
- $L_1, \ldots, L_V$ are the corresponding range sets
  - $X_v \in L_v$, $v = 1, \ldots, V$
  - Measurement Space $\mathcal{M} = \bigtimes_{i \in I} L_i$
- $\langle (I, x_1), \ldots, (I, x_Z) \mid x_z \in \mathcal{M} \rangle$ Measurement Sequence
- $\langle c_1, \ldots, c_z \rangle$ corresponding sequence of class tags
- $\{\langle (I, x_1), \ldots, (I, x_Z) \mid x_z \in \mathcal{M} \rangle, \langle c_1, \ldots, c_z \rangle\}$ Training Set
- $J_1, \ldots, J_M \subset I$ are the M index sets specifying subspaces
- $\pi_{J_m}(I, x_z) = (J_m, u_z)$, $u_z \in \bigtimes_{j \in J_m} L_j$ Projection of I onto J_m

N-tuple Method

- Tables For Each Index Set and Class
  - $Z_1 = |\{z \in [1, Z] \mid c_z = 1\}|$
  - $Z_2 = |\{z \in [1, Z] \mid c_z = 2\}|$
  - $T_{m1}(J_m, u) = |\{z \in [1, Z] \mid (J_m, u) = \pi_{J_m}(I, x_z), c_z = 1\}| / Z_1$
  - $T_{m2}(J_m, u) = |\{z \in [1, Z] \mid (J_m, u) = \pi_{J_m}(I, x_z), c_z = 2\}| / Z_2$
- Scores For Each Class
  - $S_k(I, x) = \sum_{m=1}^{M} T_{mk}(\pi_{J_m}(I, x))$
- Identification
  - Assign class k if $S_k(I, x) > S_j(I, x) + \epsilon, j \neq k$
  - Otherwise Assign reserve decision

Scanning N-tuple Classifier



S.M. Lucas and A. Amiri, *Recognition of Chain-coded Handwritten Characters With the Scanning N-Tuple Method*, **Electronics Letters**, vol. 31, no. 24, 1995, pp. 2088-2089.

Scanning N-tuple Classifier Index Sets

$$
\begin{aligned}
J_0 &= \{0, 1, 2\} \\
J_1 &= \{1, 2, 3\} \\
&\vdots \\
J_9 &= \{7, 8, 9\}
\end{aligned}
$$

Universal Approximator Conjecture

The N-tuple Subspace Classifier is a kind of universal approximator.

Conjecture

Let $\mathcal{M} = \times_{d=1}^{D} L_d$ *be the D-dimensional measurement space. Let* $f : \mathcal{M} \rightarrow \{0, 1\}$ *be a given function associating every measurement tuple with a* 0 *or a* 1*. Let P be a probability distribution on* $\mathcal{M}$*. Let* $\mathcal{T}$ *be the tables and T be the function that the n-tuple method produces to assign a class. If f is 'zzz' simple, then for every* $\epsilon > 0$*, there exists* $K << D$ *and* $M < \binom{D}{K}$ *and a two class N-tuple subspace classifier* $\mathcal{C} = (\mathcal{M}, \mathcal{J}, \mathcal{T}, K, M)$ *such that*

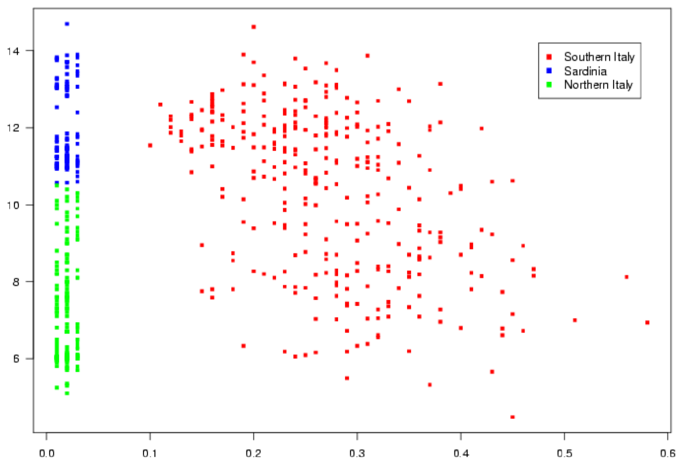
$$P(\{x \in \mathcal{M} \mid f(x) \neq T(x)\} < \epsilon$$

Where Did the Olives Come From?

- Classes
  - Northern Italy
  - Southern Italy
  - Sardinia
- Fatty Acid Measurements
  - Eicosenoic: x_1
  - Linoleic: x_2

Olives

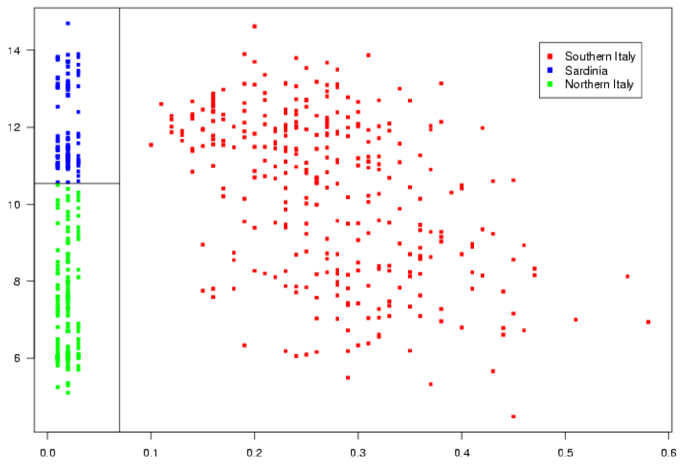
Linoleic



Eicosenoic

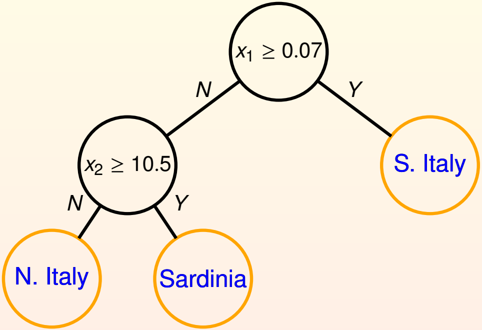
Olives

Linoleic

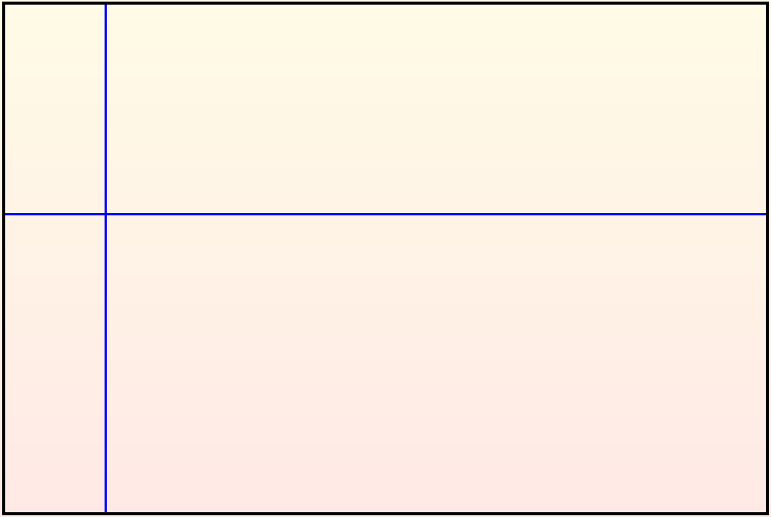


Eicosenoic

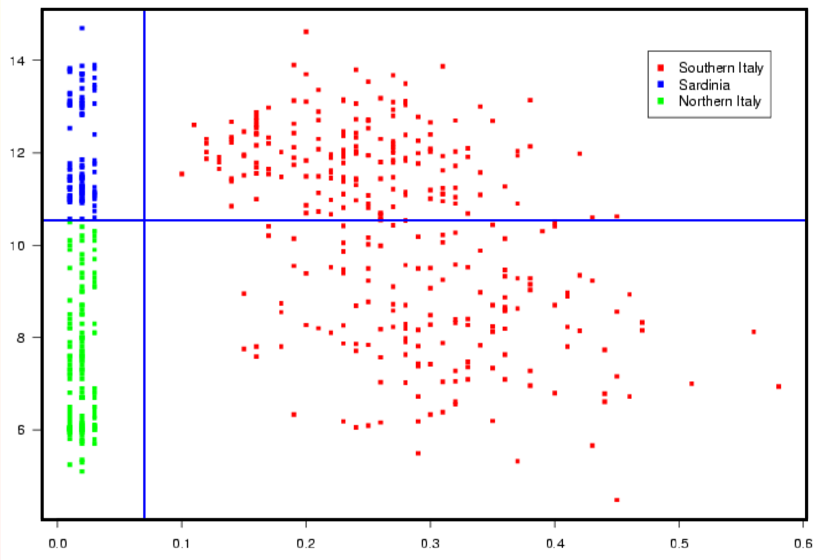
Decision Tree



Binary Quantization

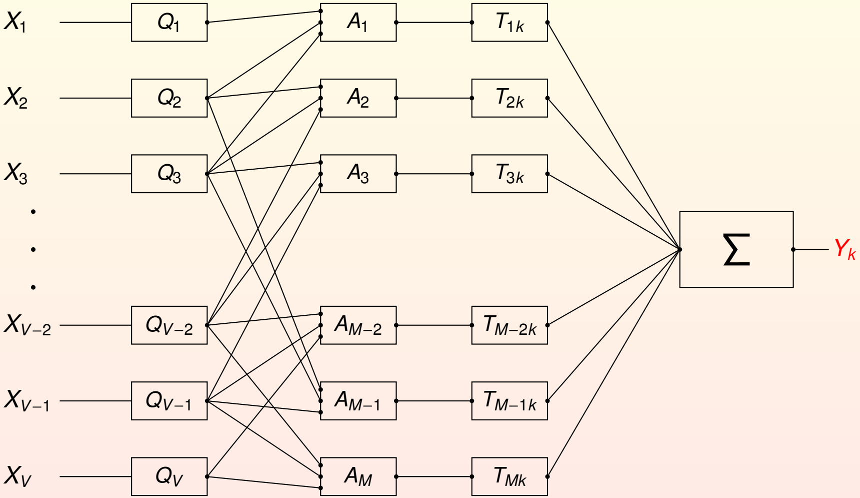


Olives



14
12
10
8
6
0.0
0.1
0.2
0.3
0.4
0.5
0.6
Southern Italy
Sardinia
Northern Italy

The N-tuple Calculation for Class k



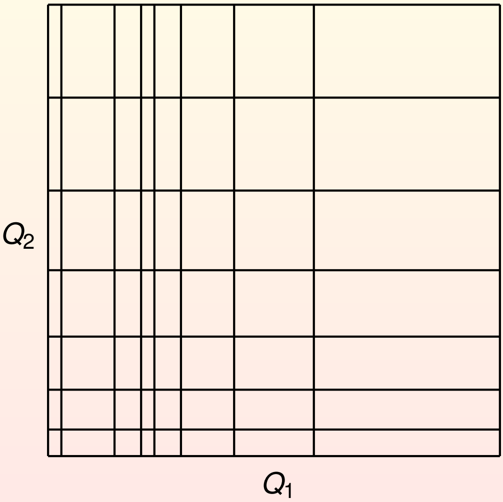
Measurement Quantizers Tuple

Address Generators

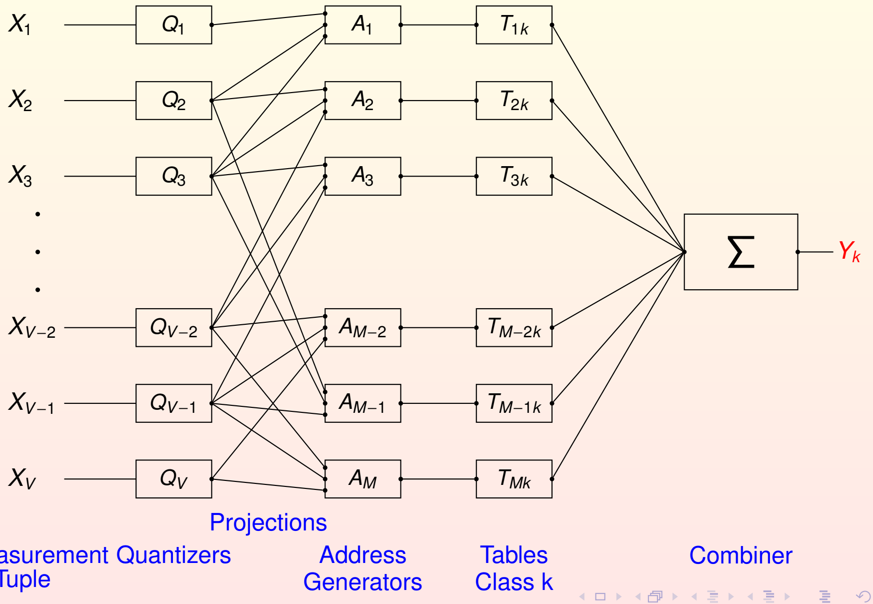
Tables Class k

Combiner

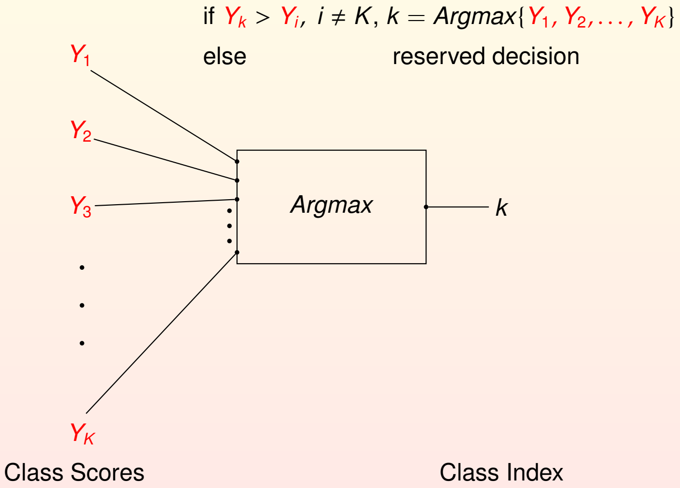
Non-uniform Quantization



The N-tuple Calculation for Class k



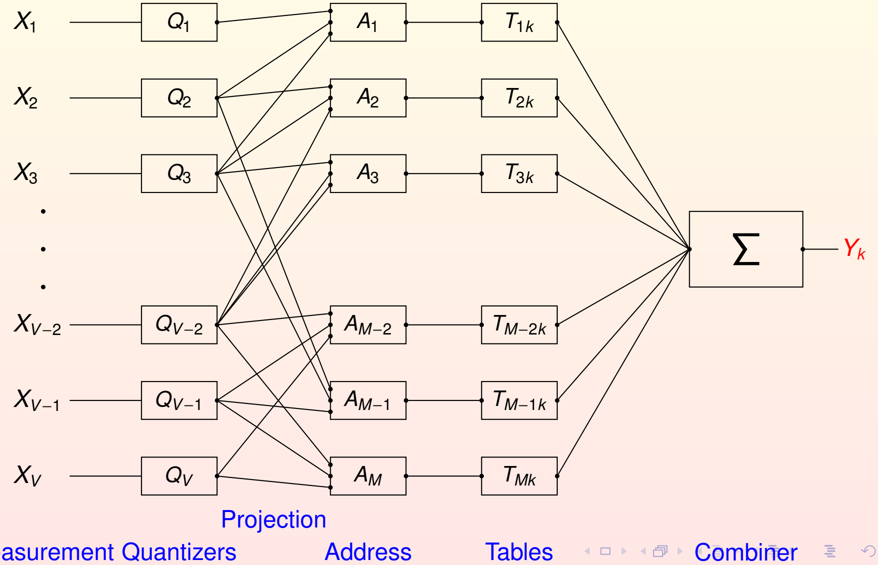
The N-tuple Class Index Generator



Optimizing the N-tuple Classifier

- Quantization
  - Optimize the number of quantized levels for each feature
  - Find the Optimal Quantizer boundaries
- Projections
  - Find the Optimal Index Sets
- Tables
  - Find the Optimal Values for all Table Entries
- Combiner
  - Find the Optimal way to Combine Scores
- Class Index
  - Optimize the way the Class Index is Determined

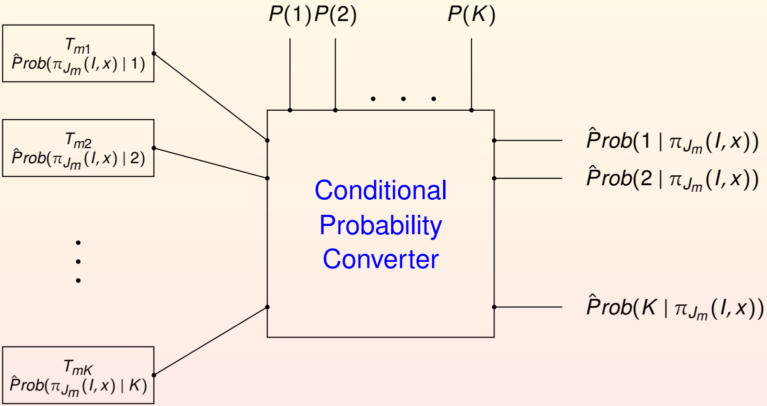
N-tuple Method Using Measurement Conditional Probabilities



$$
\begin{aligned}
T_{mk}(J_m, u) &= \hat{P}rob((J_m, u) \mid k) \\
\hat{P}rob(J_m, u) &= \sum_{k'=1}^{K} \hat{P}rob((J_m, u) \mid k')P(k') \\
\hat{P}rob(k \mid (J_m, u)) &= \frac{\hat{P}rob((J_m, u) \mid k)P(k)}{\sum_{k'=1}^{K} \hat{P}rob((J_m, u) \mid k')P(k')}
\end{aligned}
$$

$$
\begin{aligned}
T_{mk}(\pi_{J_m}(I, x)) &= \hat{P}rob(\pi_{J_m}(I, x) \mid k) \\
\hat{P}rob(k \mid \pi_{J_m}(I, x)) &= \frac{\hat{P}rob(\pi_{J_m}(I, x) \mid k)P(k)}{\sum_{k'=1}^{K} \hat{P}rob(\pi_{J_m}(I, x) \mid k')P(k')}
\end{aligned}
$$

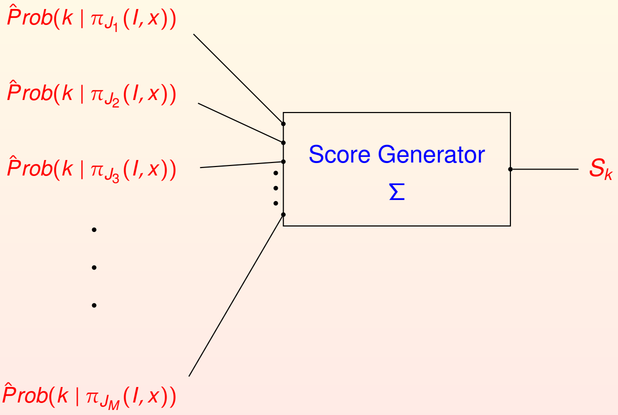
Conditional Probability Converter



$$T_{mk}(\pi_{J_m}(I,x)) = \hat{P}rob(k \mid \pi_{J_m}(I,x)) = \frac{\hat{P}rob(\pi_{J_m}(I,x) \mid k)P(k)}{\sum_{k'=1}^{K} \hat{P}rob(\pi_{J_m}(I,x) \mid k')P(k')}$$

Score Generator

$$S_k = \sum_{m=1}^{M} \hat{P}(k \mid \pi_{J_m}(I,x))$$



Conditional Probabilities

Class Given Projected Measurement

Class Score

Maximizing Expected Economic Gain: Bayes Case

A Bayes rule can always be implemented as a deterministic decision rule

			ASSIGNED CLASS			
			1	2		K
T	1	$P_T(1,d)$	$e(1,1)$	$e(1,2)$		$e(1,K)$
R	2	$P_T(2,d)$	$e(2,1)$	$e(2,2)$		$e(2,K)$
U					$\cdots$	
E					$\vdots$	
	K	$P_T(K,d)$	$e(K,1)$	$e(K,2)$		$e(K,K)$

$$\sum_{j=1}^{K} e(j,k)P_T(j,d)$$

$P_T(j,d)$ is the fraction of instances that a d from the training set has true class j

Assign any class k to d such that $\sum_{j=1}^{K} e(j,k)P_T(j,d)$ is maximal

Maximizing Expected Economic Gain: Subspace Case

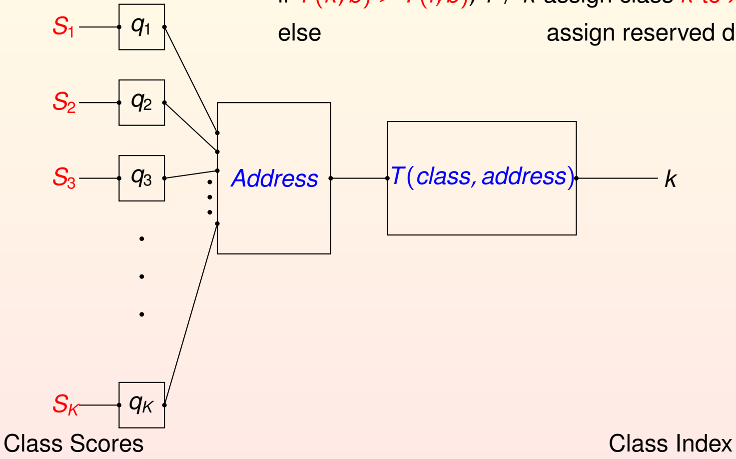
- Training Set: $\{\langle x_1, \ldots, x_z \rangle, \langle c_1, \ldots, c_Z \rangle\}$
- Tuple x_z produces K scores $S_1(x_z), \ldots, S_K(x_z)$
- The scores are quantized
  - $q_k : R \rightarrow L_k = \{0, 1, \ldots, P_k\}$, $k = 1, \ldots, K$
  - $q_1(S_1(x_z)), \ldots, q_K(S_K(x_z))$
- The quantized score produces an address
  - $a(q_1(S_1(x_z)), \ldots, q_K(S_K(x_z)))$
- The address enables us to define the table T
  - $T(k, b) = \frac{|\{z \in [1, Z] \mid b = a(q_1(S_1(x_z)), \ldots, q_K(S_K(x_z))), c_z = k\}|}{Z}$
  - $T(k, b)$ is the fraction of instances that an x_z from the training sequence has address b and class k
  - $(T(1, b), T(2, b) \ldots T(K, b))$ are the probabilities that an x that produces address b will have true classes $(1, 2, \ldots, K)$
- Assign any class k to an x that produces address b such that $\sum_{j=1}^{K} e(j, k) T(j, b)$ is maximal

Economic Gain Matrix Is the Identity

Tuple x produces address b

if $T(k,b) > T(i,b),\ i \neq k$ assign class k to x

else assign reserved decision



Group Theory and Decision Problems

Dehn in 1911 articulated the following fundamental group Decision Problems

- Word Problem
- Conjugacy Problem
- Isomorphism Problem

Haralick et. al. describes a successful machine learning approach related to Whitehead minimal words in free groups.

Robert M. Haralick, Alex D. Miasnikov, and Alexei G. Myasnikov, *Pattern Recognition and Minimal Words In Free Groups of Rank 2*, **Journal of Group Theory**, Vol. 8, 2005, 523-538.

Conjugacy Problem

Definition

Given a group G and elements u and $v \in G$, Determine if there is an element $a \in G$ such that

$$u = ava^{-1}$$

Non-Abelian Infinite Groups Tested

- Three Non-virtually Nilpotent Polycyclic Groups
  - $\mathcal{O} \rtimes U_{14}$ determined by $x^9 - 7x^3 - 1$
  - $\mathcal{O} \rtimes U_{16}$ determined by $x^{11} - x^3 - 1$
  - $\mathcal{O} \rtimes U_{34}$ determined by $x^{23} - x^3 - 1$
- Two Non-polycyclic Metabelian Groups
  - Baumslag-Solitar Group BS(1,2)
  - Generalized Metabelian Baumslag-Solitar group GMBS(2,3)
- A Non-Solvable Linear Group
  - $\mathrm{SL}(2, \mathbb{Z})$

In these infinite groups the conjugacy problem is undecidable.

Jonathan Gryak, Robert Haralick, and Delaram Kahrobaei, *Solving the Conjugacy Decision Problem Via Machine Learning*, *Experimental Mathematics*, Vol 29, Number 1, 2020, pp. 66-78

Undecidability

Definition

A family of problems with Yes/No answers is Undecidable if and only if there is no algorithm that terminates with the correct answer for every problem in the family.

Definition

The Halting Problem is to determine whether there exists an algorithm that takes a computer program and the computer program's input and decides whether it eventually halts instead of entering an infinite loop.

The halting problem is undecidable.

Protocols

- Machine Learning Techniques
  - Decision Trees: Scikit-learn DecisionTreeClassifier
  - Random Forests: Scikit-learn Random ForestClassifier
  - N-Tuple Neural Network: Our own Python Implementation
- Data Generation
  - Three Independent Data Sets For Each Group
    - Training
    - Optimization
    - Verification
  - 20,000 Geodesic word pairs
    - 10,000 conjugate pairs
    - 10,000 non conjugate pairs

Best Performing Decision Tree Classifiers

Group	Split Criterion	Depth	Accuracy
BS(1,2)	Entropy	Depth Limit	92.00%
$\mathcal{O} \rtimes U_{14}$	Entropy	Depth Limit	98.49%
$\mathcal{O} \rtimes U_{16}$	Entropy	No Depth Limit	97.23%
$\mathcal{O} \rtimes U_{34}$	Entropy	Depth Limit	98.47%
GMBS(2,3)	Gini Impurity	Depth Limit	95.43%
$\mathrm{SL}(2,\mathbb{Z})$	Entropy	No Depth Limit	96.26%

Best Performing Random Forest Classifiers for All Groups

Group	Split Criterion	Depth	Accuracy
BS(1,2)	Entropy	No Depth Limit	93.64%
$\mathcal{O} \rtimes U_{14}$	Entropy	No Depth Limit	98.69%
$\mathcal{O} \rtimes U_{16}$	Entropy	Depth Limit	98.19%
$\mathcal{O} \rtimes U_{34}$	Entropy	No Depth Limit	98.89%
GMBS(2,3)	Entropy	No Depth Limit	96.49%
SL$(2, \mathbb{Z})$	Entropy	No Depth Limit	97.47%

Best Performing N-Tuple Classifiers for All Groups

Group	Number of Subspaces	Size of Subspaces	Accuracy
BS(1,2)	30	4	92.41% (log)
$O \rtimes U_{14}$	20	3	98.77% (log)
$O \rtimes U_{16}$	20	5	98.46% (Σ)
$O \rtimes U_{34}$	100	3	99.50% (log)
GMBS(2,3)	30	4	96.13% (Σ)
$SL(2,\mathbb{Z})$	50	4	99.81% (log)

Comparison

Group	Decision Tree	Random Forest	N-tuple
BS(1,2)	92.00%	93.64%	92.41%
$O \rtimes U_{14}$	98.49%	98.69%	98.77%
$O \rtimes U_{16}$	97.23%	98.19%	98.46%
$O \rtimes U_{34}$	98.47%	98.89%	99.50%
GMBS(2,3)	95.43%	96.49%	96.13%
$SL(2,\mathbb{Z})$	96.26%	97.47%	99.81%

N-Tuple: Accuracy by Class for Tested Groups

	Accuracy by Class	
Group	Conjugate	Non-Conjugate
BS(1,2)	88.17%	96.64%
$\mathcal{O} \rtimes U_{14}$	99.95%	97.58%
$\mathcal{O} \rtimes U_{16}$	99.50%	97.41%
$\mathcal{O} \rtimes U_{34}$	99.14%	99.86%
GMBS(2,3)	97.37%	94.88%
SL$(2, \mathbb{Z})$	99.87%	99.75%