COMPUTER GRAPHICS AND IMAGE PROCESSING **13**, 191–221 (1980)

Using Perspective Transformations in Scene Analysis

ROBERT M. HARALICK

Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

Received August 25, 1978

This paper briefly reviews in a self-contained manner the perspective transformation governing 2-D images taken of a 3-D world. Using coordinates of points, direction cosines of lines on the image, or 3-D models of objects, relationships are developed that permit determination of the perspective transformation parameters in closed form, as well as the 3-D coordinates of objects. *Contents*. I, Introduction; I.1, 3-D Scene Analysis. II, The Perspective Transformation in 2-D. III, The Perspective Transformation in 3-D. IV, Properties of the 3-D Perspective Transformation; IV.1, Lines to Lines; IV.2, Vanishing Points. V, Image Coordinates to 3-D Coordinates; V.1, Camera Geometry; V.2, Vanishing Point to Camera Parameters; V.3, The Inverse Projective Transformation; V.4, Using the Inverse Perspective Transformation; V.5, Using the Perspective Transformation; V.6, Models. VI, Example; VI.1, Example 1; VI.2, Example 2. Acknowledgement. References.

I. INTRODUCTION

The perspective transformation and its mathematical properties are well known. Coexter [1] gives a treatment of it mathematically and Wolf [7] illustrates its use in photogrammetric engineering and surveying applications. Duda and Hart [3] discuss its use in scene analysis. D'Amelia [2] gives a good treatment of perspective drawing for those people interested in understanding perspectives in a visual manner without equations.

It is the purpose of this paper to review briefly and in a self-contained manner the perspective transformation and some of its important properties. Then we develop a variety of relationships based on the perspective transformation which are helpful for deducing 3-D location information from the 2-D information on the image. The contribution of the paper is in the reference value of the collection of these relationships in one place for those researchers in scene analysis who do not have the assorted reports or appendixes of these which also might develop some of them.

I.1. 3-D Scene Analysis

We are often interested in interpreting the information on a 2-D image of a 3-D world in order to determine the placement of the 3-D objects portrayed in the image. To do this we need to understand the perspective transformation governing the way 3-D information is translated onto the 2-D image. Thus, we first review the basic and well-known concepts and properties of this perspective transformation. Then we show how partial knowledge about distances between points or parallel lines in the 3-D world can be used with their perspective projection on the image in order to determine the location of these points or lines in the 3-D world.

In Sections II and IV we develop the perspective transformation, first in a 2-D world and then in a 3-D world. In Section IV we show how the perspective transformation equation forces lines in the 3-D world to transform to lines in the image. Then we show why one end of parallel lines in the 3-D world must converge

0146-664X/80/070191-31$02.00/0

to a point, called the vanishing point, on the image. We give the general vanishing point equation, as well as the specialization of the equation for lines parallel to the axes of the original coordinate system.

In Section V we discuss some of the ways perspective information on the image can be used to determine the location of points or lines in the 3-D world. We show how knowledge of one or more vanishing points on the image can help determine the camera parameters. Then we develop the inverse perspective transformation and show how knowledge of the relative positions of points in the 3-D world can be used with the coordinates of their perspective projections to determine their actual 3-D coordinates. Finally, we show how the direction cosines of lines on the image relate to their corresponding lines in the 3-D world. Specializing this relationship to lines parallel to one of the axes lying in the x, y, or $z = k$ plane, we give the equations which relate the parameter k to the camera parameters and direction cosines of the parallel lines. Finally, we specialize this relationship to perpendicular lines.

The paper concludes with an example illustrating the use of the techniques of Section V. The example image, taken with a 35-mm camera having a 50-mm lens, portrays two parallel lines of known distance apart and lying in a $z = k$ plane. In addition, there are two points in the $z = k$ plane of known distance apart in the 3-D world. The example applies the relationships developed in Section V to determine the camera parameters. Then with the camera parameters known, relationships such as the distance in the 3-D world between any two points lying on a vertical line can be determined from the coordinates of those points on the image.

II. THE PERSPECTIVE TRANSFORMATION IN 2-D

In this section we suppose we have a camera taking one-dimensional pictures in a two-dimensional world. As shown in Fig. 1, we assume that the camera lens is at the origin and points directly down the y axis. In order to keep the image in a positive orientation, we assume that the image line is at a distance f in front of the camera lens and that the lens projects forward to it. This eliminates the problems of left–right reversal in an image behind the lens. The image line for this first example is parallel to the x axis.

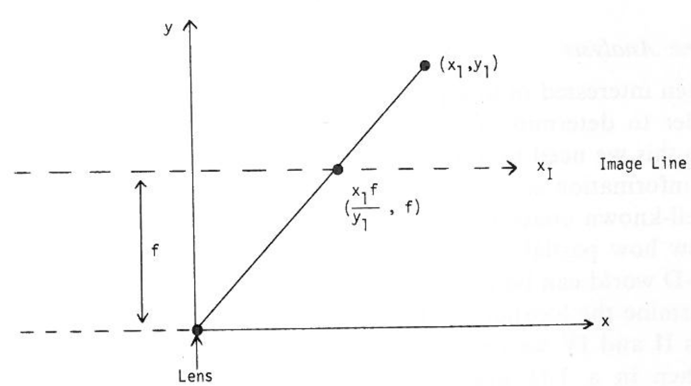

FIG. 1. How for a lens oriented along the y axis and an image plane parallel to the x axis, the perspective projection gives the point (x_1, y_1) the coordinates $(x_1/y_1, f)$ on the one-dimensional image.

According to the geometric ray optics model for the lens, the lens will focus a point (r, s) onto the image line which is a line parallel to the x axis and at a distance f directly in front of the lens. The position in the line is determined by where the line from (r, s) to origin intersects the image line. Hence, the perspective projection has coordinates $((r \cdot f)/s, f)$ in the original two-dimensional coordinate system. Relative to the one-dimensional coordinate system of the image line, the coordinate is $(r \cdot f)/s$.

Note that both the numerator and denominator of $(r \cdot f)/s$ are linear combinations of r and s. This suggests that if the numerator and denominator were computed by an appropriate linear transformation, the one-dimensional perspective coordinates could be computed by taking ratios of components of the transformed vector. We illustrate this using homogeneous coordinates. The point (r, s) can be represented as $(r, s, 1)$ in the homogeneous coordinate system. The first linear transformation translates the point $(r, s, 1)$ down the y axis by a distance f. The second transformation takes the perspective transformation to the image line. Hence,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/f & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} r \\ s \\ 1 \end{pmatrix}.$$

The one-dimensional image line coordinates for the point are then given by

$$x_I = \frac{u}{v} = \frac{rf}{s}.$$

Figure 2 illustrates a more complex example. The lens is still at the origin, but is pointing down the y' axis. The x'–y' axes are the x–y axes rotated by an angle of θ. The projection (r, s) of the point (p, q) can be determined as the intersection of the image line with the line from the origin to (p, q). The image line is given by the equation,

$$(-\sin\theta \;\; \cos\theta)\begin{pmatrix} x \\ y \end{pmatrix} = f.$$

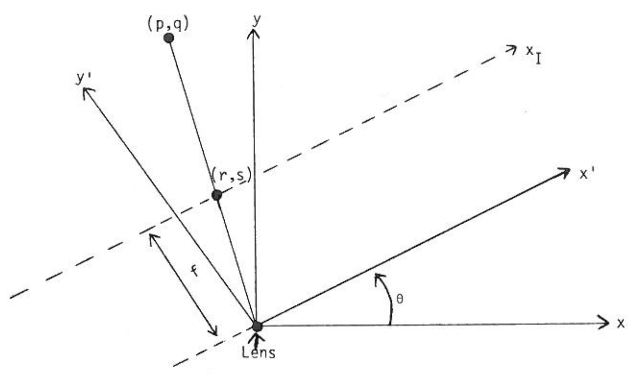

FIG. 2. The geometry of a general one-dimensional perspective projection in a two-dimensional world. The lens is at the origin and looks down the y' axis. The image line is a distance f in front of the lens and it is parallel to the x' axis. The x'–y' axes are the x–y axes rotated clockwise by an angle θ.

The line between the origin and the point (p, q) is given by

$$(-q p)\begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

Hence the perspective projection (r, s) is determined by

$$\begin{pmatrix} -\sin\theta & \cos\theta \\ -q & p \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

This system of equations has the solution,

$$\begin{pmatrix} r \\ s \end{pmatrix} = \left(\frac{f}{-p\sin\theta + q\cos\theta}\right)\begin{pmatrix} p \\ q \end{pmatrix}.$$

To represent this in a coordinate system relative to the image line, we must first rotate the x–y axes to the x'–y' axes. Let us call the coordinates of (r, s) in the x'–y' reference frame (r', s'). Then

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} r' \\ s' \end{pmatrix}.$$

Hence,

$$\begin{bmatrix} r' \\ s' \end{bmatrix} = \frac{f}{-p\sin\theta + q\cos\theta}\begin{bmatrix} p\cos\theta + q\sin\theta \\ -p\sin\theta + q\cos\theta \end{bmatrix}$$

$$= f\begin{bmatrix} \dfrac{p\cos\theta + q\sin\theta}{-p\sin\theta + q\cos\theta} \\ 1 \end{bmatrix}.$$

Note that the y' coordinate is f, exactly as we expect it to be, since the image line is at a distance f down the y' axis from the origin. The x' coordinate is the coordinate with respect to the image line. Hence,

$$x_I = f\frac{p\cos\theta + q\sin\theta}{-p\sin\theta + q\cos\theta}.$$

This relationship also has a numerator and denominator which are linear combinations of the original coordinates (p, q). Hence, the point x_I can be written as the ratio of two linear combinations.

Rewriting the relationships in terms of a homogeneous coordinate system, we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/f & 1 \end{pmatrix}}_{\text{Perspective projection}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Translation to image line}} \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Rotation of axes}} \begin{bmatrix} p \\ q \\ 1 \end{bmatrix}.$$

Dividing u by v there results x_I:

$$x_I = \frac{u}{v} = f\frac{p\cos\theta + q\sin\theta}{-p\sin\theta + q\cos\theta}.$$

III. THE PERSPECTIVE TRANSFORMATION IN 3-D

The perspective transformation in 3-D parallels the general perspective transformation in 2-D. To obtain the image frame coordinates for a given point in 3-D space, we first translate this point to a 3-D coordinate system centered at the lens of the camera. Then we rotate the coordinate system so that its x–z plane is parallel to the desired image x–z plane. Finally, the coordinates in the image are then obtained by translating the rotated coordinate system along its y axis to the desired location of the image and taking the perspective transformation to it.

We do this using a homogeneous coordinate system, assuming an arbitrary position and orientation of the lens. Let

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

be the original coordinates of a point in 3-D space and

$$\begin{pmatrix} x' \\ z' \end{pmatrix}$$

be the coordinates of the perspective projection of

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

in the image. Let

$$\begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix}$$

be the position of the lens in the original coordinate system. Assume the lens is pointing down the y axis in a new coordinate system obtained by rotating the x–y plane through an angle θ, rotating the y–z plane through an angle ϕ, and rotating the x–z plane through an angle Ψ. θ is called the camera pan angle, ϕ is called the tilt angle, and Ψ is called the swing angle. First we consider the case where $\Psi = 0$. Figure 3 illustrates the geometry of the situation. We obtain from Fig. 4 the perspective transformation:

$$\begin{pmatrix} x' \\ z' \end{pmatrix} = \frac{f}{-(x - x_I)\sin\theta\cos\phi + (y - y_I)\cos\theta\cos\phi + (z - z_I)\sin\phi} \times \begin{bmatrix} (x - x_I)\cos\theta + (y - y_I)\sin\theta \\ (x - x_I)\sin\theta\sin\phi \\ -(y - y_I)\cos\theta\sin\phi \\ +(z - z_I)\cos\phi \end{bmatrix}. \tag{1}$$

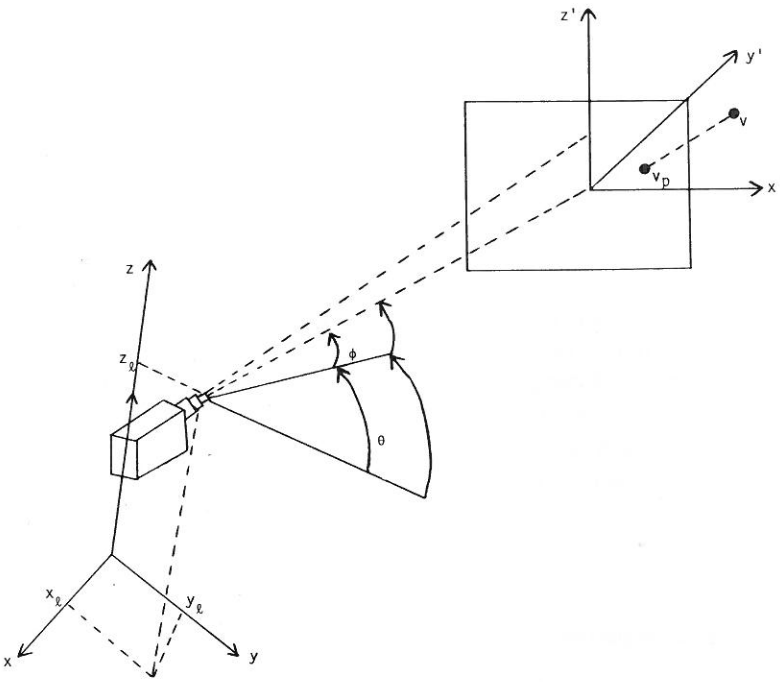

FIG. 3. The reference frame for camera and the reference frame for the perspective image plane when $\Psi = 0$.

Perspective Projection · Translation to Image · Rotation by θ and ϕ · Translation to Center of Lens

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{f} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -f \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\cos\phi \sin\theta & \cos\phi \cos\theta & \sin\phi & 0 \\ \sin\phi \sin\theta & -\sin\phi \cos\theta & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -x_\ell \\ 0 & 1 & 0 & -y_\ell \\ 0 & 0 & 1 & -z_\ell \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x^* \\ z^* \\ t^* \end{pmatrix}$$

and $x' = \frac{x^*}{t^*}$; $z' = \frac{z^*}{t^*}$.

Then $$x' = f\frac{(x - x_\ell)\cos\theta + (y - y_\ell)\sin\theta}{-(x - x_\ell)\cos\phi \sin\theta + (y - y_\ell)\cos\phi\cos\theta + (z - z_\ell)\sin\phi}$$

$$z' = f\frac{(x - x_\ell)\sin\phi\sin\theta - (y - y_\ell)\sin\phi\cos\theta + (z - z_\ell)\cos\phi}{-(x - x_\ell)\cos\phi\sin\theta + (y - y_\ell)\cos\phi\cos\theta + (z - z_\ell)\sin\phi}$$

FIG. 4. How the homogeneous coordinate representation can be used to obtain a 3-D perspective transformation. First the original coordinate system is translated to the center of the lens. Then the x–y axes are rotated by θ and the y–z axes are rotated by ϕ. These coordinates are translated down to the image plane and the perspective is taken to it.

To determine the new representation after rotating the x–z axes by Ψ, let $\begin{pmatrix} x'' \\ y'' \end{pmatrix}$ be the coordinates after rotation by Ψ. Then

$$\begin{pmatrix} x'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\Psi & \sin\Psi \\ -\sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} x' \\ z' \end{pmatrix}. \tag{2}$$

IV. PROPERTIES OF THE 3-D PERSPECTIVE TRANSFORMATION

The two most important properties of the 3-D perspective transformation is the way lines in the 3-D world relate to lines in the image plane. They are: (1) lines in the 3-D world transform to lines in the image plane and (2) parallel lines in the 3-D world meet in a vanishing point in the image plane. In this section, we give the basis for these properties and derive the equations which relate the positions and directions of lines in the 3-D world to their slopes in the perspective image plane.

IV.1. Lines to Lines

Lines in the original 3-D world correspond to lines in the image plane. To see this, let p_1 and p_2 be two points in the 3-D world. The line passing through p_1 and p_2 consists of all points having the form $\lambda p_1 + (1 - \lambda)p_2$ for some constant λ. Let the perspective projection of p_1 and p_2 be

$$\begin{pmatrix} u_1/w_1 \\ v_1/w_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2/w_2 \\ v_2/w_2 \end{pmatrix},$$

where for some matrix T,

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = Tp_1 \quad \text{and} \quad \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = Tp_2.$$

The exact nature of the matrix T is not important in this discussion. We only need to use the fact that T is a linear operator. Hence,

$$\begin{aligned} T(\lambda p_1 + (1 - \lambda)p_2) &= \lambda Tp_1 + (1 - \lambda)Tp_2 \\ &= \lambda \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} + (1 - \lambda) \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix}. \end{aligned}$$

The line in the image plane passing through

$$\begin{pmatrix} u_1/w_1 \\ v_1/w_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_2/w_2 \\ v_2/w_2 \end{pmatrix}$$

consists of all points of the form

$$\eta \begin{pmatrix} u_1/w_1 \\ v_1/w_1 \end{pmatrix} + (1 - \eta) \begin{pmatrix} u_2/w_2 \\ v_2/w_2 \end{pmatrix},$$

for some constant η. To show that the perspective transformation of every line in the 3-D world is a line in the image plane and every line in the image plane corresponds to a line in the 3-D model, we need to show that for every λ there exists an η and for every η there exists a λ such that the following relationship is satisfied:

$$\begin{bmatrix} \dfrac{\lambda u_1 + (1-\lambda)u_2}{\lambda w_1 + (1-\lambda)w_2} \\ \dfrac{\lambda v_1 + (1-\lambda)v_2}{\lambda w_1 + (1-\lambda)w_2} \end{bmatrix} = \eta \begin{bmatrix} \dfrac{u_1}{w_1} \\ \dfrac{v_1}{w_1} \end{bmatrix} + (1-\eta) \begin{bmatrix} \dfrac{u_2}{w_2} \\ \dfrac{v_2}{w_2} \end{bmatrix}.$$

It is easily verified that if λ is given, then

$$\eta = \frac{\lambda w_1}{\lambda w_1 + (1-\lambda)w_2}.$$

And if η is given, then

$$\lambda = \frac{\eta w_2}{\eta w_2 + (1-\eta)w_1}$$

make the required relationship satisfied. Therefore, lines correspond to lines.

IV.2. Vanishing Points

In this section we assume that the rotation angle $\Psi = 0$. We represent lines in the 3-D world by their direction cosines. Thus, a line consists of the set of points:

$$\left\{ \begin{pmatrix} x \\ y \\ x \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ for some constant } \lambda \right\}.$$

The vector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

is a vector having unit length ($a^2 + b^2 + c^2 = 1$) and is the direction cosines of the line. We say that two lines are parallel if they have the same direction cosines.

Two parallel lines in the 3-D world will converge to one point on the image. This point is called a vanishing point. To see this, substitute

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

for x, y, and z in the equation given for the image coordinates (x', z') in terms of the 3-D world coordinates (x,y,z) (see Fig. 4). We obtain

$$x' = f\frac{(x_0 + \lambda a - x_l)\cos\theta + (y_0 + \lambda b - y_l)\sin\theta}{-(x_0 + \lambda a - x_l)\cos\phi\sin\theta + (y_0 + \lambda b - y_l)\cos\phi\cos\theta + (z_0 + \lambda c - z_l)\sin\phi},$$

$$z' = f\frac{(x_0 + \lambda a - x_l)\sin\phi\sin\theta - (y_0 + \lambda b - y_l)\sin\phi\cos\theta + (z_0 + \lambda c - z_l)\cos\phi}{-(x_0 + \lambda a - x_l)\cos\phi\sin\theta + (y_0 + \lambda b - y_l)\cos\phi\cos\theta + (z_0 + \lambda c - z_l)\sin\phi}.$$

Now take the limit as λ goes either to $+\infty$ or $-\infty$. We obtain

$$x' = f\frac{a\cos\theta + b\sin\theta}{-a\cos\phi\sin\theta + b\cos\phi\cos\theta + c\sin\phi}, \tag{3}$$

$$z' = f\frac{a\sin\phi\sin\theta - b\sin\phi\cos\theta + c\cos\phi}{-a\cos\phi\sin\theta + b\cos\phi\cos\theta + c\sin\phi}. \tag{4}$$

Note that x' and z' depend only on the direction cosines of the line and the orientation of the new axes relative to the old ones. There is no dependence on the point

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

through which the line passes. Hence, parallel lines in the 3-D world will converge to one point on the image.

Some care is required for the physical interpretation of why both ends of the line seem to converge to the same image coordinates since the limit as λ approaches $+\infty$ and $-\infty$ is the same. Refer to Fig. 5, which illustrates the situation for a two-dimensional perspective. Equations (3) and (4) say that the perspective projection of the line ends is independent of any point through which the line may pass. Hence, the given line L and the line L', which is the line L translated so that it passes through the origin, have the same vanishing point as shown in figure. Let us suppose, without loss of generality, that the sense of the direction cosine vector is such that $\lambda \to +\infty$ brings us to the top end or limit of the line and that $\lambda \to -\infty$ brings us to the bottom limit of the line. Note that the top limit of the line is in front of the lens so that as $\lambda \to +\infty$ we are discussing a situation that our camera can see. The bottom limit of the line is behind the lens so that as $\lambda \to -\infty$ we are discussing a situation that our camera cannot see. The distinction about what the camera can see and cannot see is purely a physical interpretation of the geometrical model. For if we interpreted the image plane to be behind the lens rather than in front of the lens, then the vanishing point shown in Fig. 5 would correspond to the $\lambda \to -\infty$ limit of the line.

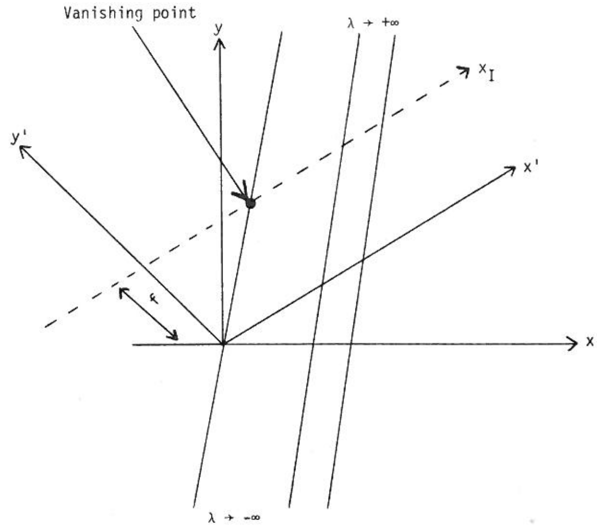

FIG. 5. The vanishing point for all lines in the parallel set shown.

It is easy to tell which limit of a line is in front of the lens. Let

$$\begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

be a unit length vector of the direction cosines indicating in what direction the lens is pointing. Let the direction cosines of a line be

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

If the dot product

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

is positive, then $\lambda \to +\infty$ is the limit of the line in front of the lens. If the dot product is negative, then $\lambda \to -\infty$ is the limit of the line in front of the lens and $\lambda \to +\infty$ is the limit of the line behind the lens.

Equation (4) reduces to the especially simple form $z' = 0$ for all lines in some $z = k$ (direction cosines of the form $(a, b, 0)$) plane when the camera tilt angle ϕ is zero. Because in an outdoor picture, taken with zero tilt angle, the horizon appears as the $z' = 0$ line, the $z' = 0$ line is called the horizon line. This means that vanishing points for parallel horizontal lines must lie on the horizon line.

There are three vanishing points of particular interest and they correspond to lines parallel to the original x–y–z axes. The direction cosine for lines parallel to the x axis (horizontal lines) is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and the vanishing point for all such lines is

$$\begin{pmatrix} x_H \\ z_H \end{pmatrix} = f \begin{bmatrix} \dfrac{-1}{\cos\phi \tan\theta} \\ -\tan\phi \end{bmatrix}. \tag{5}$$

The direction cosine for lines parallel to the y axis (depth lines) is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and the vanishing point for all such lines is

$$\begin{pmatrix} x_D \\ z_D \end{pmatrix} = f \begin{bmatrix} \dfrac{\tan\theta}{\cos\phi} \\ -\tan\phi \end{bmatrix}. \tag{6}$$

Note that these equations imply $z_H = z_D$ and $-x_D/x_H = \tan^2\theta$. The direction cosine for lines parallel to the z axis (vertical lines) is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the vanishing point for all such lines is

$$\begin{pmatrix} x_V \\ z_V \end{pmatrix} = f \begin{pmatrix} 0 \\ \cot\phi \end{pmatrix}. \tag{7}$$

Note that Eqs. (5) and (7) imply that

$$f = \sqrt{|z_H z_V|}\ .$$

V. IMAGE COORDINATES TO 3-D COORDINATES

The 3-D part of the analysis of a 2-D image of a 3-D world seeks to answer the following question: Where does each point on the image come from in the 3-D world? To determine the 3-D coordinates corresponding to any perspective projection point on the image, the camera parameters must first be determined. Then the 3-D coordinates can be obtained by the inverse perspective transformation in conjunction with some additional information.

In this section we show how to use the lens focal length equation and the coordinates of the observed vanishing point(s) to determine the camera parameters. Then we develop the inverse perspective transformation and show how knowledge of the relative position of points in the 3-D world can be used in conjunction with their perspective projection to determine their actual coordinates. Finally, we show how the direction cosines of lines on the image relate to their corresponding lines in the 3-D world. Specializing this relationship to lines parallel to one of the axes and lying in an x, y, or $z = k$ plane, we give the equations which relate the parameter k to the camera parameters and direction cosines of the line on the image.

V.1. Camera Geometry

One of the camera parameters is the distance f that the image is in front of the lens. This parameter can be determined from elementary camera geometry, which is illustrated in Fig. 6. To do this we assume that the focal length F of the camera lens is known, the size I of the image is known, and the size N of the negative which the camera originally created is known. Let D be the distance behind the lens that the film is located. The focal length equation is

$$\frac{1}{D} + \frac{1}{f} = \frac{1}{F}. \tag{8}$$

Letting M be the magnification, we have

$$M = \frac{I}{N} = \frac{f}{D}.$$

Substituting $D = f/M$ into the focal length equation, we obtain

$$\frac{M}{f} + \frac{1}{f} = \frac{1}{F}.$$

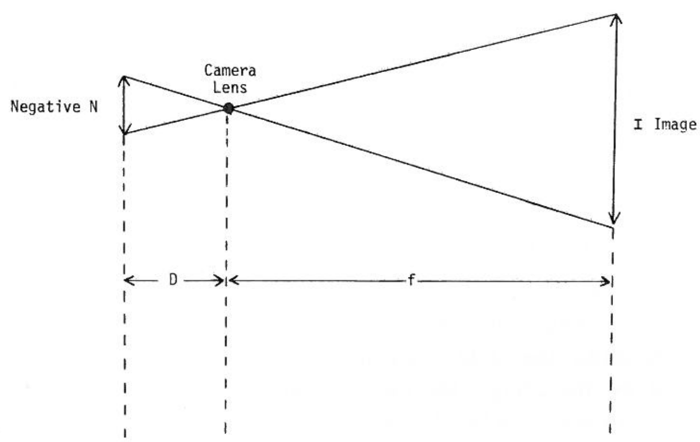

FIG. 6. The camera geometry for a lens of focal length F. The focal length equation is: $1/D + 1/f = 1/F$.

Solving this equation for f yields

$$f = (M + 1)F. \tag{9}$$

Hence, the magnification of the image with respect to the film and the focal length of the camera lens can determine the distance f in front of the lens which the image is located.

V.2. Vanishing Point to Camera Parameters

For convenience, we take the origin of the original 3-D coordinate system to be the center of the camera lens. Following our convention, we let f be the distance in front of the lens that the perspective x–z image plane is located. Let the pan and tilt angles of the camera be θ and ϕ, respectively, and the x–z rotation angle be Ψ. We initially assume that $\Psi = 0$ and then we generalize some of our results.

First, suppose that f is known. If the horizontal vanishing point is known, then from Eq. (5),

$$\tan\phi = -z_{\mathrm{H}}/f \tag{10}$$

and

$$\tan\theta = z_{\mathrm{H}}/(x_{\mathrm{H}}\sin\phi). \tag{11}$$

If the depth vanishing point is known, then from Eq. (6),

$$\tan\phi = -z_{\mathrm{D}}/f \tag{12}$$

and

$$\tan\theta = (x_{\mathrm{D}}/f)\cos\phi. \tag{13}$$

If the vertical vanishing point is known, then from Eq. (7), only the angle ϕ can be determined:

$$\tan\phi = f/z_{\mathrm{V}}. \tag{14}$$

Next, suppose that f is not known, that Ψ is not guaranteed to be zero, but both the horizontal (x_H, z_H) and depth (x_D, z_D) vanishing points are known. Then by rotating the x-z axes by $-\Psi$ we can obtain vanishing points for an image whose swing angle is 0. The vanishing points in the rotated axes are given by

$$\begin{pmatrix} x'_{\mathrm{H}} \\ z'_{\mathrm{H}} \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} x_{\mathrm{H}} \\ z_{\mathrm{H}} \end{pmatrix}$$
$$\text{and} \begin{pmatrix} x'_{\mathrm{D}} \\ z'_{\mathrm{D}} \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} x_{\mathrm{D}} \\ z_{\mathrm{D}} \end{pmatrix}. \tag{15}$$

Hence, by Eqs. (10) and (12),

$$\tan\phi = -\frac{x_{\mathrm{H}}\sin\Psi + z_{\mathrm{H}}\cos\Psi}{f} = -\frac{x_{\mathrm{D}}\sin\Psi + z_{\mathrm{D}}\cos\Psi}{f}.$$

Solving this equation for Ψ, we obtain

$$\tan \Psi = \frac{z_D - z_H}{x_H - x_D}. \tag{16}$$

Once Ψ is known, f, θ, and ϕ can be determined using the rotated vanishing points (x'_H, z'_H) and (x'_D, z'_D) from the rotated axes. From Eqs. (11) and (13),

$$\tan \theta = \frac{z'_H}{x'_H \sin \phi} = \frac{x'_D}{f} \cos \phi. \tag{17}$$

Hence,

$$f = \frac{x'_D x'_H}{z'_H} \cos \phi \sin \phi.$$

But by Eq. (10), $f = -z'_H/\tan \phi = (z'_H \cos \phi)/\sin \phi$, and putting these two equations together, we obtain

$$\sin \phi = z'_H \sqrt{\frac{1}{|x'_D x'_H|}} . \tag{18}$$

With ϕ known, Eq. (17) gives

$$\tan \theta = \frac{z'_H}{x'_H \sin \phi} = \operatorname{sign}(x'_H) \sqrt{\left| \frac{x'_D}{x'_H} \right|} . \tag{19}$$

When the vertical vanishing point is known, then Ψ can be determined by noticing that from Eq. (7):

$$\begin{pmatrix} x_V \\ z_V \end{pmatrix} = \begin{pmatrix} -\sin \Psi f \cot \phi \\ \cos \Psi f \cot \phi \end{pmatrix}.$$

Hence,

$$\tan \Psi = -x_V / z_V. \tag{20}$$

When the horizontal and vertical vanishing points are known, the distance f can be determined using Eqs. (5) and (7):

$$f = \sqrt{|(x_H \sin \Psi + z_H \cos \Psi)(x_V \sin \Psi + z_V \cos \Psi)|} . \tag{21a}$$

Similarly, when the depth and vertical vanishing points are known, the distance f can be determined using Eqs. (6) and (7):

$$f = \sqrt{|(x_D \sin \Psi + z_D \cos \Psi)(x_V \sin \Psi + z_V \cos \Psi)|} . \tag{21b}$$

V.3. The Inverse Projective Transformation

Each point (x', z') on the image can be the perspective projection of many points in the 3-D world. All of the points having perspective projection (x', z') must be located on a line emanating out of (x', z') and passing through the lens. It is the purpose of this section to give a representation for that line so that in the next section we can use that representation in order to develop some of the relationships between points in the 3-D world and points on the image.

For convenience, we take the origin of the original 3-D coordinate system to be the center of the camera lens. Following our convention, we let f be the distance in front of the lens that the perspective x–z image plane is located. Then with respect to the rotated reference frame whose origin is at the lens, the point (x', z') on the image is the point (x', f, z') in the 3-D world. Since we desire our representation to be in terms of the original reference frames, our first problem is to express the point (x', f, z') in terms of the original coordinate system.

Suppose that the pan, tilt, and swing angles of the camera are θ, ϕ, and Ψ, respectively. Then the rotated reference frame is obtained from the original reference frame by rotating the x–y plane by θ and then rotating the y–z plane by ϕ and then rotating the x–z plane by Ψ. To express (x', f, z') in terms of the original reference frame, we just need to undo these rotations. First, we have to rotate the x–z plane by $-\Psi$, then rotate the y–z plane by $-\phi$, and then the x–z plane by $-\theta$.

With respect to the original coordinate system, let (u, v, w) be the coordinates of the point (x', f, z'). Then

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\Psi & 0 & -\sin\Psi \\ 0 & 1 & 0 \\ \sin\Psi & 0 & \cos\Psi \end{bmatrix} \begin{bmatrix} x' \\ f \\ z' \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\Psi + \sin\theta\sin\phi\sin\Psi & -\sin\theta\cos\phi & \sin\theta\sin\phi\cos\Psi - \cos\theta\sin\Psi \\ \sin\theta\cos\Psi - \cos\theta\sin\phi\sin\Psi & +\cos\theta\cos\phi & -\cos\theta\sin\phi\cos\Psi - \sin\theta\sin\Psi \\ \cos\phi\sin\Psi & \sin\phi & \cos\phi\cos\Psi \end{bmatrix} \begin{bmatrix} x' \\ f \\ z' \end{bmatrix}. \quad (22)$$

Since the lens is the origin, a line passing through the lens and the point (u, v, w) consists of all multiples of (u, v, w). Hence, the line whose perspective projection in the image is (x', z') consists of the points

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix} \text{ for some constant } \lambda \right\}.$$

To simplify our further analysis, we assume that the angle $\Psi = 0$. We may do this without loss of generality since if (x'', z'') is the point on an image whose x–z

rotation is Ψ, then by setting

$$\begin{pmatrix} x' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix}\begin{pmatrix} x'' \\ z'' \end{pmatrix},$$

we obtain the coordinates (x', z') of the point on an image whose x–z rotation angle Ψ is zero. With this assumption, the line whose perspective projection in the image is (x', z') consists of the points

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x'\cos\theta - f\sin\theta\cos\phi + z'\sin\theta\sin\phi \\ x'\sin\theta + f\cos\theta\cos\phi - z'\cos\theta\sin\phi \\ f\sin\phi + z'\cos\phi \end{pmatrix} \text{for some } \lambda \right\}.$$

V.4. Using the Inverse Perspective Transformation

The inverse projective transformation gives us a representation of 3-D points in terms of 2-D points. This representation is useful in a number of ways. In this section we assume that the swing angle Ψ is zero and that the camera lens is the origin. We discuss:

(1) how certain useful planes in the 3-D world project to a line on the image;

(2) how the perspective projection (x', z') of a point (x, y, z) can be used to determine two of its coordinates when a third coordinate is known;

(3) how the perspective projections of two points known to have a common but unknown value in one coordinate can be used to determine the value of the common coordinate when the relative positions of the points in the 3-D world are known; and

(4) how the perspective projections of three points known to have a common but unknown value in one coordinate can be used to determine the camera parameters f, θ, and ϕ as well as the actual coordinates of the points when only their relative positions are known.

The horizontal line on the image which is the perspective projection of the plane of eye-level points is easily determined. Since the camera lens is the origin, eye-level points have z coordinates which are zero. Hence,

$$0 = \lambda(f\sin\phi + z'\cos\phi) \qquad \text{for all } \lambda.$$

This implies that

$$z' = \frac{-f\sin\phi}{\cos\phi} = -f\tan\phi. \tag{23}$$

Therefore, all eye-level points in the 3-D world will have a perspective projection whose z' coordinate is $-f\tan\phi$.

Likewise, the perspective projection of the plane $x = 0$ is a line on the image consisting of the points,

$$\left\{ \begin{pmatrix} x' \\ z' \end{pmatrix} \middle| 0 = x' \cos\theta - f \sin\theta \cos\phi + z' \sin\theta \sin\phi \right\}, \tag{24}$$

and the perspective projection of the plane $y = 0$ is a line on the image consisting of the points,

$$\left\{ \begin{pmatrix} x' \\ z' \end{pmatrix} \middle| 0 = x' \sin\theta + f \cos\theta \cos\phi - z' \cos\theta \sin\phi \right\}. \tag{25}$$

Another use of the inverse perspective transformation is for using the perspective projection (x', z') of a point (x, y, z) to determine two of its coordinates when the third is known. For example, suppose the z coordinate of a point whose perspective projection is (x', z') is known. If $f \sin\phi + z' \cos\phi \neq 0$ so that $z \neq 0$, it is possible to determine the 3-D x and y coordinates of the point from (x', z'). Set

$$z = \lambda(f \sin\phi + z' \cos\phi)$$

and solve this equation for λ.

$$\lambda = \frac{z}{f \sin\phi + z' \cos\phi}.$$

Now substitute this expression back in the equations for x and y. We obtain

$$x = z \frac{x' \cos\theta - f \sin\theta \cos\phi + z' \sin\theta \sin\phi}{f \sin\phi + z' \cos\phi}, \tag{26}$$

$$y = z \frac{x' \sin\theta + f \cos\theta \cos\phi - z' \cos\theta \sin\phi}{f \sin\phi + z' \cos\phi}. \tag{27}$$

In a similar manner, if the y coordinate is known and nonzero, we obtain

$$x = y \frac{x' \cos\theta - f \sin\theta \cos\phi + z' \sin\theta \sin\phi}{x' \sin\theta + f \cos\theta \cos\phi - z' \cos\theta \sin\phi}, \tag{28}$$

$$z = y \frac{f \sin\phi + z' \cos\phi}{x' \sin\theta + f \cos\theta \cos\phi - z' \cos\theta \sin\phi}. \tag{29}$$

And if the x coordinate is known and nonzero, then

$$y = x \frac{x' \sin\theta + f \cos\theta \cos\phi - z' \cos\theta \sin\phi}{x' \cos\theta - f \sin\theta \cos\phi + z' \sin\theta \sin\phi}, \tag{30}$$

$$z = x \frac{f \sin\phi + z' \cos\phi}{x' \cos\theta - f \sin\theta \cos\phi + z' \sin\theta \sin\phi}. \tag{31}$$

If two points in the 3-D world are known to have a common value in one coordinate, these equations allow us to determine that common value if we know the difference between some other coordinate of these points. For example, if (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points satisfying $z_1 = z_2 = z$ and whose perspective projections are (x_1', z_1') and (x_2', z_2'), and if the difference $x_1 - x_2$ is known, then

$$x_1 - x_2 = z\left[\frac{x_1'\cos\theta - f\sin\theta\cos\phi + z_1'\sin\theta\sin\phi}{f\sin\phi + z_1'\cos\phi} - \frac{x_2'\cos\theta - f\sin\theta\cos\phi + z_2'\sin\theta\sin\phi}{f\sin\phi + z_2'\cos\phi}\right].$$

If $(x_1 - x_2) \neq 0$, then by collecting terms and solving for z by dividing by an expression guaranteed to be nonzero, it follows that

$$z = \frac{(x_1 - x_2)(f\sin\phi + z_1'\cos\phi)(f\sin\phi + z_2'\cos\phi)}{(x_1' - x_2')f\cos\theta\sin\phi + (z_1' - z_2')f\sin\theta + (x_1'z_2' - x_2'z_1')\cos\theta\cos\phi}. \tag{32}$$

If the difference $y_1 - y_2$ is known and nonzero, then

$$z = \frac{(y_1 - y_2)(f\sin\phi + z_1'\cos\phi)(f\sin\phi + z_2'\cos\phi)}{(x_1' - x_2')f\sin\phi\sin\theta - (z_1' - z_2')f\cos\theta + (x_1'z_2' - x_2'z_1')\sin\theta\cos\phi}. \tag{33}$$

If the coordinate having common value is y and the difference $x_1 - x_2$ is known and nonzero, then

$$y = \frac{(x_1 - x_2)(x_1'\sin\theta + f\cos\theta\cos\phi - z_1'\cos\theta\sin\phi)\cdot(x_2'\sin\theta + f\cos\theta\cos\phi - z_2'\cos\theta\sin\phi)}{(x_1' - x_2')f\cos\phi + (x_2'z_1' - x_1'z_2')\sin\phi}. \tag{34}$$

If the difference $z_1 - z_2$ is known and nonzero, then

$$y = \frac{(z_1 - z_2)(x_1'\sin\theta + f\cos\theta\cos\phi - z_1'\cos\theta\sin\phi)\cdot(x_2'\sin\theta + f\cos\theta\cos\theta - z_2'\cos\theta\sin\phi)}{-(x_1' - x_2')f\sin\theta\sin\phi + (z_1' - z_2')f\cos\theta + (-x_1'z_2' + x_2'z_1')\sin\theta\cos\phi}. \tag{35}$$

If the coordinate having common value is x and the difference $y_1 - y_2$ is known and nonzero, then

$$x = \frac{(y_1 - y_2)(x_1'\cos\theta - f\sin\theta\cos\phi + z_1'\sin\theta\sin\phi)\cdot(x_2'\cos\theta - f\sin\theta\cos\phi + z_2'\sin\theta\sin\phi)}{-(x_1' - x_2')f\cos\phi + (x_1'z_2' - x_2'z_1')\sin\phi}. \tag{36}$$

If the difference $z_1 - z_2$ is known and nonzero, then

$$x = \frac{(z_1 - z_2)(x_1'\cos\theta - f\sin\theta\cos\phi + z_1'\sin\theta\sin\theta)\cdot(x_2'\cos\theta - f\sin\theta\cos\phi + z_2'\sin\theta\sin\phi)}{-(x_1' - x_2')f\cos\theta\sin\phi - (z_1' - z_2')f\sin\theta - (x_1'z_2' - x_2'z_1')\cos\theta\cos\phi}. \tag{37}$$

Thus, we have shown that when f, θ, and ϕ are known and the lens is the origin, then the perspective projections of two points which have known and nonzero difference in one coordinate and unknown but common value in another coordinate, comprise sufficient information to recover the values for each of the coordinates of the points. The recovery is achieved by using one of the second set of equations (Eqs. (32)–(37)) to determine the common coordinate value and then using two appropriate equations from the first set of equations (Eqs. (26)–(31) to determine the remaining coordinate values for each of the two points.

We close this section by showing how the relative positions of three points to one another in a $z = k$ plane, where k is unknown, is sufficient to determine the actual coordinates of the three points, as well as the camera parameters f, θ, and ϕ.

Suppose

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

are three points with common z coordinates so that $z_1 = z_2 = z_3$. Suppose further than $x_1 \neq x_2 \neq x_3$ and $y_1 \neq y_2 \neq y_3$. Then

$$\begin{aligned} &\frac{(x_1 - x_2)(f\sin\phi + z_1'\cos\phi)(f\sin\phi + z_2'\cos\phi)}{(x_1' - x_2')f\cos\theta\sin\phi + (z_1' - z_2')f\sin\theta + (x_1'z_2' - x_2'z_1')\cos\theta\cos\phi} \\ &= \frac{(y_1 - y_2)(f\sin\phi + z_1'\cos\phi)(f\sin\phi + z_2'\cos\phi)}{(x_1' - x_2')f\sin\phi\sin\theta - (z_1' - z_2')f\cos\theta + (x_1'z_2' - x_2'z_1')\sin\theta\cos\phi} \\ &= \frac{(x_2 - x_3)(f\sin\phi + z_2'\cos\phi)(f\sin\phi + z_3'\cos\phi)}{(x_2' - x_3')f\cos\theta\sin\phi + (z_2' - z_3')f\sin\theta + (x_2'z_3' - x_3'z_2')\cos\theta\cos\phi} \\ &= \frac{(y_2 - y_3)(f\sin\phi + z_2'\cos\phi)(f\sin\phi + z_3'\cos\phi)}{(x_2' - x_3')f\sin\phi\sin\theta - (z_2' - z_3')f\cos\theta + (x_2'z_3' - x_3'z_2')\sin\theta\cos\phi}. \end{aligned} \tag{38}$$

These three equations allow the determination of the parameters f, θ, and ϕ. (Unfortunately, these equations are highly nonlinear.) Nevertheless, once f, θ, and ϕ are known, the common value for the z coordinate can be determined as before and from it the x and y coordinates can be determined. This means that knowledge of the relative positions of three points on a $z = k$ plane where k is unknown is sufficient to determine the camera parameters f, θ, and ϕ and from them the coordinates of each of the points.

V.5. Using the Perspective Transformation

Two points in the 3-D world determine a line. As shown in Section IV.1, this line in the 3-D world projects to a line in the image. In this section we determine the relationship between the parameters of the line in the 3-D world and the direction cosines of the line in the image. We do this using the perspective transformation of Section III assuming that the swing angle Ψ is zero and that the camera lens is the origin.

Recall that if (u, v, w) is a point in the 3-D world and (x, z) is the coordinates of the perspective projection of (u, v, w), the perspective projection (x, z) is given by

$$\begin{pmatrix} x \\ z \end{pmatrix} = \frac{f}{-u\cos\phi\sin\theta + v\cos\phi\cos\theta + w\sin\phi} \times \begin{bmatrix} u\cos\theta + v\sin\theta \\ u\sin\theta\sin\phi - v\cos\theta\sin\phi + w\cos\phi \end{bmatrix}. \tag{39}$$

We represent the line in the 3-D world in its direction cosine form. A line passing through the point (x_0, y_0, z_0) and having direction cosines (a, b, c) consists of the points,

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ for some constant } \lambda \right\}.$$

Let (x_1, z_1) and (x_2, z_2) be the perspective projection of the points $(x_0, y_0, z_0) + \lambda_1(a, b, c)$ and $(x_0, y_0, z_0) + \lambda_2(a, b, c)$, respectively. Then the direction cosines (m, n) of the line on the image satisfy

$$(x_1 - x_2)n = (z_1 - z_2)m. \tag{40}$$

Substituting $(x_0, y_0, z_0) + \lambda_1(a, b, c)$ and $(x_0, y_0, z_0) + \lambda_2(a, b, c)$ into the perspective projection equation (1), and determining x_1, x_2, z_1, and z_2, we obtain

$$[\cos\phi(ay_0 - bx_0) + \cos\theta\sin\phi(az_0 - cx_0) + \sin\theta\sin\phi(bz_0 - cy_0)]n = [\sin\theta(az_0 - cx_0) + \cos\theta(bz_0 - cy_0)]m.$$

Upon rearranging this equation there results:

$$\begin{aligned} &x_0[-nb\cos\phi - nc\cos\theta\sin\phi + mc\sin\theta] \\ &\quad + y_0[na\cos\phi - nc\sin\theta\sin\phi + mc\cos\theta] \\ &\quad + z_0[na\cos\theta\sin\phi + nb\sin\theta\sin\phi - mb\cos\theta - ma\sin\theta] = 0. \end{aligned} \tag{41}$$

This equation is useful in a number of ways. For example, if the direction cosines of a line in the image can be measured and if two out of the three coordinates (x_0, y_0, z_0) are known, then Eq. (41) allows the determination of the third coordinate. Perhaps more interesting is Eq. (41) specialized for lines which are parallel to the x or y or z axes. The direction cosines for all lines parallel to x axis is $(a, b, c) = (1, 0, 0)$. For these lines,

$$y_0[n\cos\phi] + z_0[n\cos\theta\sin\phi - m\sin\theta] = 0. \tag{42}$$

For lines parallel to the y axis, $(a, b, c) = (0, 1, 0)$ and

$$x_0[-n\cos\phi] + z_0[n\sin\theta\sin\phi - m\cos\theta] = 0. \tag{43}$$

For lines parallel to the z axis, $(a, b, c) = (0, 0, 1)$ and

$$x_0[-n\cos\theta\sin\phi + m\sin\theta] + y_0[-n\sin\theta\sin\phi + m\cos\theta] = 0. \tag{44}$$

These equations imply that for lines parallel to the axes, knowledge of one of the coordinates of the point (x_0, y_0, z_0) is sufficient to determine one of the other coordinates. These equations also allow relative position information of two parallel lines to determine one of the coordinates of the point (x_0, y_0, z_0). For example, let $(x_1, y_1, z_1) + \lambda(1, 0, 0)$ and $(x_2, y_2, z_2) + \eta(1, 0, 0)$ be two lines parallel to the x axis. Suppose that these lines lie in the plane $z = z_0$. Hence $z_1 = z_2 = z_0$. Let the direction cosines of the perspective projection of these lines be (m_1, n_1) and (m_2, n_2), respectively. Then, from Eq. (42),

$$y_1 n_1 \cos\phi + z_0(n_1\cos\theta\sin\phi - m_1\sin\theta) = 0$$

and

$$y_2 n_2 \cos\phi + z_0(n_2\cos\theta\sin\phi - m_2\sin\theta) = 0.$$

Solving this pair of equations for z_0 in terms of $(y_1 - y_2)$ we obtain

$$z_0 = \frac{(y_1 - y_2)n_1 n_2 \cos\phi}{(m_1 n_2 - m_2 n_1)\sin\theta}. \tag{45}$$

Hence, the common z coordinates for these two parallel lines is proportional to the difference in their y coordinates. Knowledge of the difference of the y coordinates is sufficient to determine the common z coordinate.

In a similar manner, if the lines $(x_1, y_1, z_1) + \lambda(1, 0, 0)$ and $(x_2, y_2, z_2) + \eta(1, 0, 0)$, which are parallel to the x axis, lie in the plane $y = y_0$, then

$$y_0 = \frac{(z_1 - z_2)(-n_1\cos\theta\sin\phi + m_1\sin\theta)(-n_2\cos\theta\sin\phi + m_2\sin\theta)}{(m_2 n_1 - m_1 n_2)\sin\theta\cos\phi}. \tag{46}$$

If the lines are $(x_1, y_1, z_1) + \lambda(0, 1, 0)$ and $(x_2, y_2, z_2) + \eta(0, 1, 0)$ (parallel to the y axis) lie in the plane $x = x_0$, then

$$x_0 = \frac{(z_1 - z_2)(+n_1\sin\theta\sin\phi - m_1\cos\theta)(n_2\sin\theta\sin\phi - m_2\cos\theta)}{(m_1 n_2 - m_2 n_1)\cos\theta\cos\phi}. \tag{47}$$

If these lines lie in the plane $z = z_0$, then

$$z_0 = \frac{(x_1 - x_2)\cos\phi\, n_1 n_2}{(m_2 n_1 - m_1 n_2)\cos\theta}. \tag{48}$$

If the lines are $(x_1, y_1, z_1) + \lambda(0, 0, 1)$ and $(x_2, y_2, z_2) + \eta(0, 0, 1)$ (parallel to the z axis) lie in the plane $y = y_0$, then

$$y_0 = \frac{(x_1 - x_2)(n_1\cos\theta\sin\phi - m_1\sin\theta)(n_2\cos\theta\sin\phi - m_2\sin\theta)}{(m_2 n_1 - m_1 n_2)\sin\phi(\sin^2\theta - \cos^2\theta)}. \tag{49}$$

If these lines lie in the plane $x = x_0$, then

$$x_0 = \frac{(y_1 - y_2)(n_1 \sin\theta \sin\phi - m_1 \cos\theta)(n_2 \sin\theta \sin\phi - m_2 \cos\theta)}{(m_2 n_1 - m_1 n_2)\sin\phi(\cos^2\theta - \sin^2\theta)}. \tag{50}$$

Hence, knowledge of the distance between two parallel lines lying in a x, y, or $z = k$ plane and knowledge of their direction cosines on the image is sufficient to determine the parameter k of the plane.

If the rotation angle Ψ is 0, then if three edges which are known to be mutually orthogonal meet at coordinates (x_0, y_0, z_0), then the direction cosines of these lines on the image are sufficient to determine a relationship between the angles θ and ϕ. We assume here that the camera rotation angles θ, ϕ, and Ψ are relative to a coordinate system whose axes are parallel to those edges known to be mutually orthogonal.

Let (m_1, n_1), (m, n_2), and (m_3, n_3) be the direction cosines on the image of the three orthogonal lines passing through the point (x_0, y_0, z_0). By Eq. (42), (43), and (44),

$$\begin{bmatrix} 0 & n_1 \cos\phi & n_1 \cos\theta \sin\phi - m_1 \sin\theta \\ -n_2 \cos\phi & 0 & n_2 \sin\theta \sin\phi - m_2 \cos\theta \\ -n_3 \cos\theta \sin\phi + m_3 \sin\theta & -n_3 \sin\theta \sin\phi + m_3 \cos\theta & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This implies that the determinant of the matrix must be zero. Setting the determinant to zero and solving for $\sin\phi$ there results

$$\sin\phi = \frac{m_3(n_1 m_2 - m_1 n_2)\sin\theta\cos\theta}{n_2(n_1 m_3 - m_1 n_3)\sin^2\theta + n_1(m_2 n_3 - n_2 m_3)\cos^2\theta}. \tag{51}$$

V.6. Models

We sometimes have knowledge about the 3-D objects appearing on a 2-D image. Man-made objects are often rectangular parallelpipeds or some other regular geometric shape. In this section we illustrate a few techniques which can be used to determine camera parameters as well as object dimensions with this kind of knowledge.

Following Roberts [5] we define a model to be the canonical form of a known 3-D object. The model consists of the ordered pair (P, R) where $P = \{p_1, \ldots, p_N\}$ is a set of three-dimensional vertices and R is a relation, $R \subseteq P \times P$, which contains all the pairs of vertices in P which could be connected by a visible edge in the 3-D object. Using a screening and searching process, it is possible to scan the image and select a set $Q = \{q_1, \ldots, q_M\}$ of two-dimensional coordinates which have a possibility of being the perspective projection of some of the points in P and determine a set S which contain pairs of points of Q which have visible edges between them on the image. Then with a tree search, the topological correspondence or matching f between the points in Q and points in P can be established. Such a correspondence must be a relation homomorphism from S to R. It satisfies $S \circ f \subseteq R$, where the composition $\circ$ is defined by $S \circ f = \{(b_1, b_2) \in P \times P \mid$ for some $(a_1, a_2) \in S$, $b_1 = f(a_1)$ and $b_2 = f(a_2)\}$. Hence, if a_1 and a_2 are a pair of points on the image which have an edge between them, then f matches them to a pair of points b_1, b_2 in P which by the model can have a visible edge between them.

Once the correspondence is established, we may let A be a $4 \times M$ matrix whose columns are the points $f(q_1), \ldots, f(q_M)$ and whose last row consists of all ones:

$$A = \begin{bmatrix} | & | & & | \\ f(q_1) & f(q_2) & \cdots & f(q_M) \\ | & | & & | \\ 1 & 1 & & 1 \end{bmatrix}.$$

A is the matrix whose columns are the coordinates of some of the points in P expressed in a homogeneous coordinate system so they have a fourth component. Let B be a matrix whose columns are the points in Q expressed in a homogeneous coordinate system.

$$B = \begin{bmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_M \\ | & | & & | \\ 1 & 1 & & 1 \end{bmatrix}.$$

Let T be the 3×4 transformation which scales the points in P to their true dimension, rotates the camera coordinate system, and takes the perspective projection to the image plane. The relationship among A, B, and T is given by

$$TA = B.$$

As done by Roberts [5], this equation can be solved for T using a standard least-squares approach. The normal equation is

$$T = BA'(AA')^{-1}$$

and the squared error of the fit is given by

$$\| B\big(I - \big(A'(AA')^{-1}A\big) \|^2.$$

If the fit is bad (high error), then the hypothesis that the observed object can be modeled by (P, R) is rejected. If the fit is good (low error), then the hypothesis that the observed object can be modeled by (P, R) is not rejected. In this case, the camera parameters and object dimensions can be obtained from T in the following manner.

The transformation T consists of an independent scaling of x–y–z in the original coordinate system followed by a translation which places the $(0,0,0)$ point of the object at new coordinates (x_0, y_0, z_0), followed by a coordinate rotation θ, ϕ, and Ψ, a translation by f down the y axis and a perspective projection to the image plane.

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/f & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -f \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & 0 \\ g & h & i & 0 \\ j & k & l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$\begin{bmatrix} a & b & c & 0 \\ g & h & i & 0 \\ j & k & l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos\Psi\cos\theta + \sin\Psi\sin\phi\sin\theta & \cos\Psi\sin\theta - \sin\Psi\sin\phi\cos\theta & \sin\Psi\cos\phi & 0 \\ -\cos\phi\sin\theta & \cos\phi\cos\theta & \sin\phi & \\ -\sin\Psi\cos\theta + \cos\Psi\sin\phi\sin\theta & -\sin\Psi\sin\theta - \cos\Psi\sin\phi\cos\theta & \cos\Psi\cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Upon multiplying these matrices we obtain

$$T = \begin{bmatrix} d_1 a & d_2 b & d_3 c & x'_0 \\ d_1 j & d_2 k & d_3 l & z'_0 \\ d_1 g/f & d_2 h/f & d_3 i/f & y'_0/f \end{bmatrix},$$

where (x'_0, y'_0, z'_0) is (x_0, y_0, z_0) expressed in the coordinate system. Recovery of the scaling and translation transform and coordinate rotation parameters can be obtained by premultiplying T.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & f \\ 0 & 1 & 0 \end{bmatrix} T = \begin{bmatrix} d_1 a & d_2 b & d_3 c & x'_0 \\ d_1 g & d_2 h & d_3 i & y'_0 \\ d_1 j & d_2 k & d_3 l & z'_0 \end{bmatrix}.$$

The rotated translation parameters x_0, y_0, and z_0 appear in the last column. Now notice that since the matrix

$$\begin{bmatrix} a & b & c \\ g & h & i \\ j & k & l \end{bmatrix}$$

is a rotation, it is an orthonormal matrix. Hence, upon taking the square root of the sum of the squared entries of the first three columns of

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & f \\ 0 & 1 & 0 \end{bmatrix} T$$

we obtain the scaling parameters d_1, d_2, and d_3.

$$\sqrt{(d_1a)^2 + (d_1g)^2 + (d_ij)^2} = \sqrt{d_1^2(a^2 + g^2 + j^2)} = d_1,$$
$$\sqrt{(d_2b)^2 + (d_2h)^2 + (d_2k)^2} = \sqrt{d_2^2(b^2 + h^2 + k^2)} = d_2,$$
$$\sqrt{(d_3c)^2 + (d_3i)^2 + (d_3l)^2} = \sqrt{d_3^2(c^2 + i^2 + l^2)} = d_3.$$

By postmultiplying by the inverse scaling transformation, there results the rotation matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & f \\ 0 & 1 & 0 \end{bmatrix} T \begin{bmatrix} d_1^{-1} & 0 & 0 \\ 0 & d_2^{-1} & 0 \\ 0 & 0 & d_3^{-1} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ j & k & l \end{bmatrix}.$$

The angles θ, ϕ, and Ψ can be obtained from simple trigonometry. The equation $i = \sin\phi$ determines the angle ϕ to within its sign. Then

$$-g/\cos\phi = \sin\theta \quad \text{and} \quad h/\cos\phi = \cos\theta$$
$$c/\cos\phi = \sin\Psi \quad \text{and} \quad l/\cos\phi = \cos\Psi$$

determine θ and Ψ exactly. The proper sign for ϕ can be determined by checking the values for θ and Ψ in one of the entries a, b, j, or k.

VI. EXAMPLES

In this section we discuss two example uses of the equations developed in the earlier sections. Our first example comes from an agricultural problem: given a close-up view of some rows of plants, determine their height. Our second example comes from robotics: given an image of a rectangular parallelpiped, determine the camera parameters and the coordinates of the corners of the rectangular parallelpiped.

VI.1. Example 1

Figure 7 illustrates an image of a couple of rows of small cotton plants. The image is taken with a 35-mm camera having a 50-mm lens. The problem is to determine the height of each cotton plant. The only additional information known is that the cotton rows are parallel with a distance between them of about 38 in. and a spacing between plants of about 7 in. We assume that whatever point on the plant we choose for its top, that point lies vertically above the base of the plant. The camera parameters f, θ, ϕ, and Ψ are all unknown.

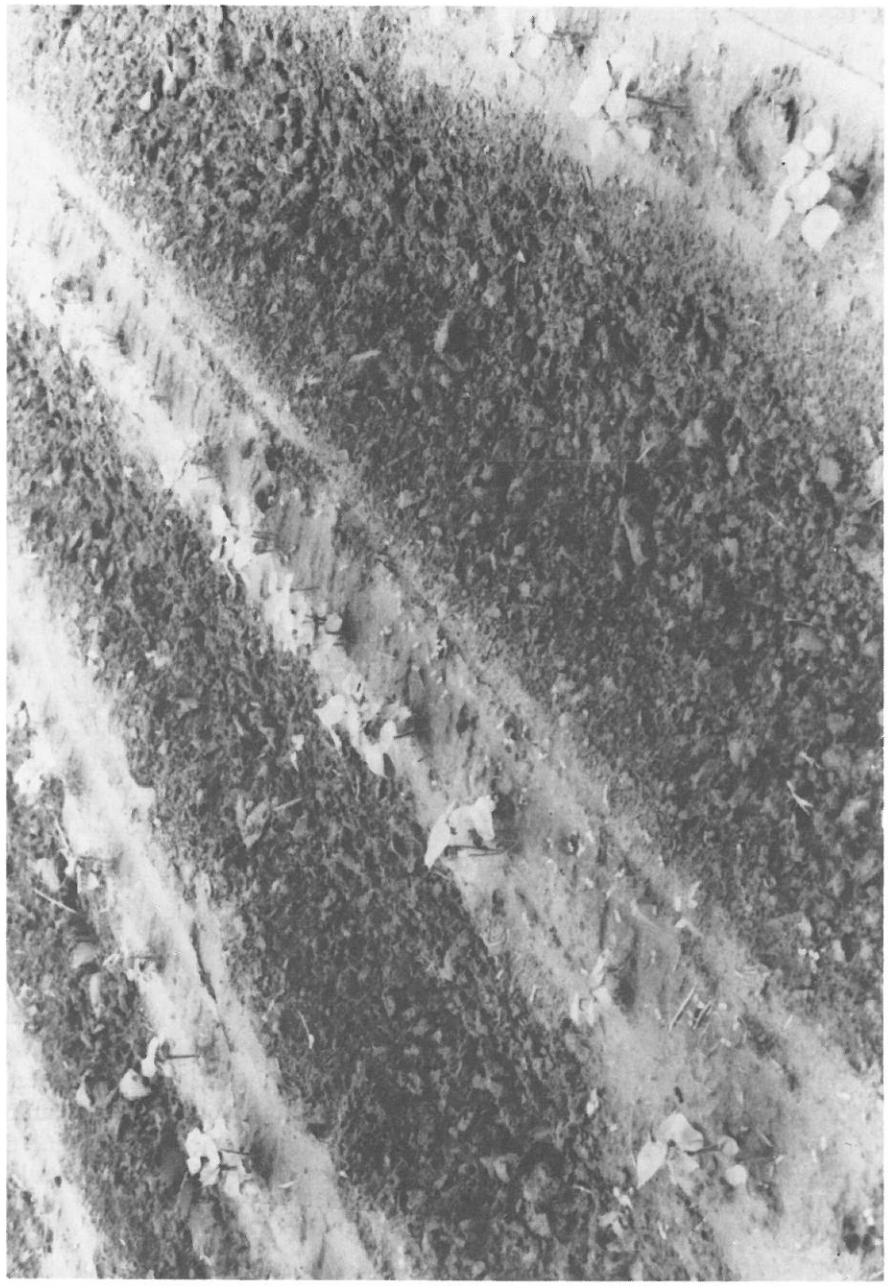

FIG. 7. An image of a couple of rows of small cotton plants.

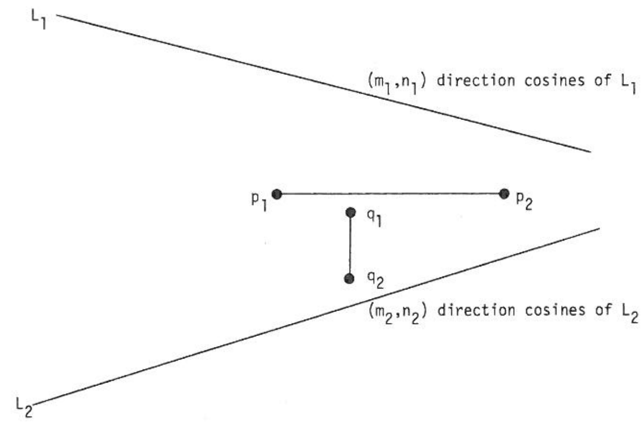

FIG. 8. The variables involved in the example image.

Figure 8 illustrates with an exaggerated perspective drawing the known and unknown information on the example image. The lines L_1 and L_2 are parallel in the 3-D world and are of known distance apart. The lines L_1 and L_2 and the points p_1, p_2, and q_1 all lie in the same $z = k$ plane, where k is unknown. The line passing through p_1 and p_2 is parallel to L_1 and L_2. In the 3-D world, the point q_2 lies vertically above the point q_1. The problem is to find the distance between q_2 and q_1.

Our analysis proceeds in the following way. The distance f which the image is in front of the lens can be easily obtained from the camera geometry. The size of a 35-mm negative is known and the size of the image is known. This determines the magnification M. The focal length F of the camera lens is known. Then, by Eq. (9),

$$f = (M + 1)F.$$

The remaining camera parameters θ, ϕ, and Ψ are more difficult to obtain. By extending the parallel lines L_1 and L_2 until they meet, we can determine the image coordinates of a vanishing point. For convenience, we take the lens as the origin of the 3-D coordinate system and take the x axis of the 3-D coordinate system to be parallel to the lines L_1 and L_2. We take the ground to be a $z = k$ plane, where k is unknown. Let the vanishing point coordinates be (x_H, z_H). Equations (10) and (11), which relate the horizontal vanishing point coordinates to the camera pan and tilt angles θ and ϕ, assume that the swing angle $\Psi = 0$. Therefore, we rotate the x–z axes by $-\Psi$ to obtain (x_H, z_H) in a coordinate system in which the x–z rotation angle is zero:

$$\begin{pmatrix} x'_H(\Psi) \\ z'_H(\Psi) \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} x_H \\ z_H \end{pmatrix}.$$

This gives the coordinates of the vanishing point in the rotated coordinate system as a function of Ψ. With f known, Eqs. (10) and (11) determine θ and ϕ.

$$\phi(\Psi) = \tan^{-1}\frac{-z'_H(\Psi)}{f},$$

$$\theta(\Psi) = \tan^{-1}\frac{z'_H(\Psi)}{x'_H(\Psi)\sin\phi(\Psi)}.$$

Since points p_1 and p_2 have the same z coordinate and lie along a line parallel to the x axis, the directed distance between p_1 and p_2 is just the difference in their x coordinates, x_1 and x_2. With $x_1 - x_2$ known, and the perspective projection of p_1 and p_2 being (x'_1, z'_1) and (x'_2, z'_2), respectively, by Eq. (32), we can solve for their common z coordinate:

$$z(\Psi) = \frac{(x_1 - x_2)[f\sin\phi(\Psi) + z'_1\cos\phi(\Psi)][f\sin\phi(\Psi) + z'_2\cos\phi(\Psi)]}{(x'_1 - x'_2)f\cos\theta(\Psi)\sin\phi(\Psi) + (z'_1 - z'_2)f\sin\theta(\Psi) + (x'_1z'_2 - x'_2z'_1)\cos\theta(\Psi)\cos\phi(\Psi)}. \quad (52)$$

The z coordinate is a function of Ψ which is still not known. However, we can use the direction cosines (m_1, n_1) and (m_2, n_2) of the lines on the image. First, we need to convert these direction cosines to those in the image whose swing angle is 0 by

$$\begin{pmatrix} m'_1(\Psi) \\ n'_1(\Psi) \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix}\begin{pmatrix} m_1 \\ n_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} m'_2(\Psi) \\ n'_2(\Psi) \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix}\begin{pmatrix} m_2 \\ n_2 \end{pmatrix}.$$

Then, using the fact that the directed distance from line L_1 to line L_2 is d_{12} and the fact that three parallel lines in the same $z = k$ plane as the points p_1 and p_2, we have by Eq. (45),

$$z(\Psi) = \frac{d_{12}\cos\phi(\Psi)n'_1(\Psi)n'_2(\Psi)}{(m'_1(\Psi)n'_2(\Psi) - m'_2(\Psi)n'_1(\Psi)\sin\theta(\Psi)}. \quad (53)$$

The two equations (52) and (53) can then be solved for the unknown Ψ. The simplest procedure is to partition the $-90°$ to $+90°$ range in which Ψ must lie into small intervals, say 5°, and do a binary search for a zero of the function which is the difference between the two expressions for the z coordinate. Once Ψ is known, then θ and ϕ, as well as z, becomes known.

Let $q_1 = (u_1, v_1, w_1)$, where w_1 is the common value for the z coordinate of the lines L_1 and L_2 and the points p_1, p_2, and q_1. Let (r_1, s_1) be the coordinates of q_1 on the image and let (r_2, s_2) be the coordinates of q_2 on the image. Express these coordinates in terms of a reference frame which is rotated by $-\Psi$. This gives

$$\begin{pmatrix} r_1' \\ s_1' \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} r_1 \\ s_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r_2' \\ s_2' \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} r_2 \\ s_2 \end{pmatrix}.$$

Now Eqs. (26) and (27) allow us to determine the x and y coordinates of the point $q_1 = (u_1, v_1, w_1)$:

$$u_1 = w_1 \frac{r_1' \cos\theta - f\sin\theta\cos\phi + s_1'\sin\theta\sin\phi}{f\sin\phi + s_1'\cos\phi}, \tag{54}$$

$$v_1 = w_1 \frac{r_1' \sin\theta + f\cos\theta\cos\phi - s_1'\cos\theta\sin\phi}{f\sin\phi + s_1'\cos\phi}. \tag{55}$$

Since the point q_2 is directly above q_1, it has coordinates $q_2 = (u_1, v_1, w_2)$. Its z coordinate w_2 can be obtained from Eqs. (29) or (31):

$$w_2 = u_1 \frac{f\sin\phi + s_2'\cos\phi}{r_2'\cos\theta - f\sin\theta\cos\phi + s_2'\sin\theta\sin\phi}, \tag{56}$$

$$= v_1 \frac{f\sin\phi + s_2'\cos\phi}{r_2'\sin\theta + f\cos\theta\cos\phi - s_2'\cos\theta\sin\phi}. \tag{57}$$

The distance which q_2 is above q_1 is then given by

$$w_2 - w_1. \tag{58}$$

VI.2. Example 2

It is interesting that if an object is only known to have a pair of orthogonal faces meeting in an edge, the camera parameters and some of the coordinates of the vertices of the meeting edge can easily be determined. For diagrammatic purposes, Fig. 9 illustrates an object in which the edges marked 1 and 1′ as well as 2 and 2′

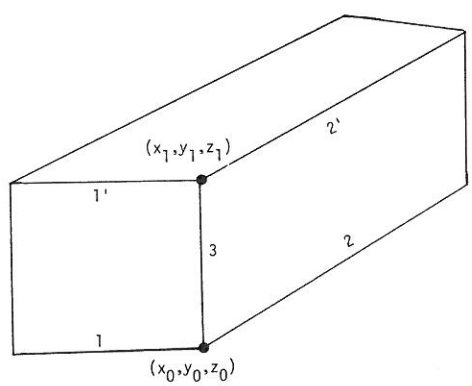

FIG. 9. A rectangular parallelpiped. The direction cosines of the edges 1, 1′, 2, and 2′ are measurable on the image and the vanishing points determined by 1 1′ and 2 2′ are measurable.

are parallel. Furthermore, the face 1 1′ is orthogonal to the face 2 2′. The faces meet in edge 3, which is orthogonal to edges 1 and 2.

Let θ, ϕ, and Ψ be the camera rotation angles relative to edges 1, 2, and 3. Let (x_H, z_H) be the coordinates of the vanishing point determined by the lines 1 1′. Let (x_D, z_D) be the coordinates of the vanishing point determined by the lines 2 2′. Then, by Eq. (16),

$$\tan\Psi = \frac{z_D - z_H}{x_H - x_D}.$$

Now undo the rotation Ψ be letting the rotated vanishing points be (x'_H, z'_H) and (x'_D, z'_D).

$$\begin{pmatrix} x'_H \\ z'_H \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi & x_H \\ \sin\Psi & \cos\Psi & z_H \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x'_D \\ z'_D \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi & x_D \\ \sin\Psi & \cos\Psi & z_D \end{pmatrix}.$$

By Eq. (18),

$$\sin\phi = z'_H \sqrt{\frac{1}{|x'_D x'_H|}}.$$

By Eq. (10),

$$f = -z'_H / \tan\phi.$$

And by Eq. (19),

$$\tan\theta = z'_H / (x'_H \sin\phi).$$

These equations completely determine the camera parameters.

The x and y coordinates of the ends (x_0, y_0, z_0) and (x_1, y_1, z_1) of edge 3 can be determined in terms of the z coordinate. Let (n_1, m_1) be the direction cosines of the line labeled 1. Let (n_2, m_2) be the direction cosines of the line labeled 2. Let (n'_1, m'_1) and (n'_2, m'_2) be the direction cosines in a coordinate system rotated by $-\Psi$. Then by Eqs. (42) and (43)

$$y_0 = \frac{-z_0[n'_1 \cos\theta \sin\phi - m'_1 \sin\theta]}{n'_1 \cos\phi},$$

$$x_0 = \frac{z_0[n'_2 \sin\theta \sin\phi - m'_2 \cos\theta]}{n'_2 \cos\phi}.$$

A similar relationship can be written for the coordinates for the lines 1′ and 2′ which meet in the corner (x_1, y_1, z_1).

ACKNOWLEDGMENTS

Support from AFOSR Grant 77-3307, and NSF Grant MCS-7923827, the encouragement of Ed Riseman to write the paper, and Lynn Ertebati's help in typing the manuscript are all gratefully acknowledged.

REFERENCES

1. H. S. M. Coexter, *The Real Projective Plane*, Cambridge Univ. Press, Cambridge, 1961.
2. J. D'Amelia, *Perspective Drawing Handbook*, Leon Amiel, New York, 1964.
3. R. Duda and P. Hart, *Pattern Classification and Scene Analysis*, Wiley, New York, 1973.
4. S. Irwin, Camera Models and Machine Perception, Stanford Artificial Intelligence Project, Memo AIM-121, Computer Science Department, Stanford University, Stanford, Calif., May 1970.
5. L. G. Roberts, Machine perception of three-dimensional solids, in *Optical and Electro-Optical Information Processing* (J. L. Trippett *et al*., Eds.), MIT Press, Cambridge, Mass., 1965.
6. J. C. C. Williams, *Simple Photogrammetry*, Academic Press, New York, 1969.
7. P. R. Wolf, *Elements of Photogrammetry*, McGraw-Hill, New York, 1974.