Two-view motion analysis: a unified algorithm

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We present a linear algorithm for determining the three-dimensional rotation and translation of a rigid object from two time-sequential perspective views using point correspondences. The algorithm is different from existing ones in two respects. First, various measures for combating noise are incorporated. Second, the algorithm is unified in the sense that, assuming that the surface assumption holds, it can handle both the case of nonzero translation and the case of zero translation.

1. INTRODUCTION

Determining the relative motion between an observer and his environment is a major problem in computer vision. Its applications include mobile robot navigation and the monitoring of dynamic industrial processes. Motion estimation also has many applications in image processing. For example, in efficient coding using digital pulse code modulation in time, motion estimation and compensation can potentially improve the compression significantly. In reducing noise in image sequences by temporal filtering, registration of the object of interest from frame to frame is necessary to avoid blurring, and registration is, in essence, equivalent to motion estimation. The reader is referred to Refs. 1 and 2 for some of these applications.

In this paper we present an approach to the determination of three-dimensional motion of a single isolated rigid body from two time-sequential perspective views (image frames).

A. Statement of the Problem

The basic geometry of the problem is sketched in Fig. 1. The object-space coordinates are denoted by lowercase letters and the image-space coordinates by uppercase letters. Specifically, consider a particular physical point on the surface of a rigid body in the scene. Let

\[
(x, y, z) = \text{object-space coordinates of the point at time } \tau_1,
\]

\[
(x', y', z') = \text{object-space coordinates of the point at time } \tau_2,
\]

\[
(X, Y) = \text{image-space coordinates of the point at time } \tau_1,
\]

\[
(X', Y') = \text{image-space coordinates of the point at time } \tau_2.
\]

It is well known in kinematics that

\[
(x', y', z')^T = R(x, y, z)^T + T,
\]

where \( R \) is a 3 x 3 orthonormal matrix of the first kind, i.e., \( R^T R = R R^T = I_3 \) (\( I_3 \) is the 3 x 3 identity matrix), \( \det(R) = 1 \).

Our problem is the following:

Given two images frames at \( \tau_1 \) and \( \tau_2 \), find the motion parameters \( T \) (to within a scale factor) and \( R \).

As we shall see later, the equations relating the motion parameters to the image-point coordinates inevitably involve the ranges (z coordinates) of the object points. Therefore, in determining the motion parameters, we also determine the ranges of the observed objects points. It will be seen that the translation vector \( T \) and the object-point ranges can be determined to within a global positive scale factor. The value of this scale factor could be found if we should know the magnitude of \( T \) or the absolute range of any observed object point.

B. A Two-Stage Approach to Solving the Problem

We present a two-stage method to solve the problem posed in Subsection 1.A. In the first stage, we find point correspondences in the two perspective views (images). By a point correspondence, we mean a pair of image coordinates \((X, Y), (X', Y')\) that are images at \( \tau_1 \) and \( \tau_2 \), respectively, of the same physical point on the object. Then, in the second stage, we determine the motion parameters from these image coordinates by solving a set of equations. This paper deals with the second stage. However, a few comments on the first stage are in order here.

In order to be able to find point correspondences, we must have images that contain points that are distinctive in some sense. For example, images of man-made objects often contain sharp corners that are relatively easy to extract. More generally, image points at which the local gray-level variations (defined in some way) are maximum can be used.

In any case, we first extract in each of the two images a large number of points that are distinctive. Then we try to match the two point patterns in the two images by using...
spatial structures of the patterns. The matching will be successful only if the amount of rotation (θ) is relatively small (so that the perspective distortion is small). For example, in Ref. 5 good matching results were obtained when θ < 5 deg. This restriction may be relaxed if we have some a priori information about the object.

C. Motion Equations

From the geometry of Fig. 1 and using Eq. (1), we can derive an equation relating the motion parameters to the coordinates of a corresponding image-point pair. Unfortunately, this equation is nonlinear. Iterative techniques for solving nonlinear equations can hardly be expected to converge to the correct solution unless a very good initial guess of the solution is available. Fortunately, by defining appropriate intermediate unknowns, it is possible to put the motion equation into a linear form. However, after these intermediate unknowns are solved for, we have to determine from the motion parameters to the correct solution unless a very good initial guess of the motion are related by Eq. (1). Taking any vector T that is collinear with T₀ and taking its cross product with both sides of Eq. (1), we obtain

\[
\begin{bmatrix} X = x/z, & Y = y/z \end{bmatrix} \Rightarrow \begin{bmatrix} X' = x'/z', & Y' = y'/z' \end{bmatrix}
\]

Recall that the 3-D coordinates of a point on the object whose three-dimensional (3-D) spatial coordinates before and after motion are \((x, y, z)\) and \((x', y', z')\), respectively. Let \((X, Y) [(X', Y')]\) be its central projective coordinates before (after) motion onto the image plane, \(z = 1\), with the projective center at the origin \(O\). The following projective equations relate the 3-D spatial coordinates and their corresponding two-dimensional projective coordinates:

\[
\begin{bmatrix} T \times (X, Y, 1) \end{bmatrix} = \begin{bmatrix} T \times [R_0(X, Y, 1)] \end{bmatrix}
\]

and, after taking dot product of both sides of Eq. (3) with \((X', Y', 1)\),

\[
(X', Y', 1)(T \times R_0)(X, Y, 1) = 0,
\]

where \(T \times R_0 = [T \times r_1, T \times r_2, T \times r_3]\); \(r_1, r_2, r_3\) being the columns of \(R_0\). Let \(E = T \times R_0\). Then Eq. (4) states that, for any image-point correspondence pair \([X, Y, (X', Y')]\), the \(3 \times 3\) matrix \(E\) satisfies the following equation that is linear and homogeneous in the elements of \(E\):

\[
(X', Y', 1)E(X, Y, 1) = 0.
\]

D. Outline of the Paper

The structure of the paper is as follows. In Section 2, the linear motion equation is derived, and a necessary and sufficient condition is given for its degeneracy. Assuming non-degeneracy, a solution to the linear motion equations is given. At this stage, we have determined the intermediate unknowns. Then, in Section 3, an algorithm is presented for finding the motion parameters from these intermediate variables. There are four candidate solutions. In Section 4, a method is described for using the image-point correspondences to pick out the unique correct solution from the four candidates. Finally, in Section 5, the entire algorithm is summarized.

Our algorithm is different from existing ones. It is probably closest in spirit to the ones given by Zhuang and Haralick13 and Yen and Huang.12 The new contributions are twofold. First, the algorithm is unified in the sense that it can handle both the case of \(T_0 \neq 0\) and the case of \(T_0 = 0\). Second, various measures are taken to combat the effect of noise in image coordinates.

2. TWO-VIEW MOTION EQUATION: GENERAL SOLUTION AND SURFACE ASSUMPTION

A. Derivation of Motion Equation

Referring to Fig. 1, we assume that a rigid body is in motion in the half-space \(z < 0\). Take a particular point on the object whose three-dimensional (3-D) spatial coordinates before and after motion are \((x, y, z)\) and \((x', y', z')\), respectively. Let \((X, Y) [(X', Y')]\) be its central projective coordinates before (after) motion onto the image plane, \(z = 1\), with the projective center at the origin \(O\). The following projective equations relate the 3-D spatial coordinates and their corresponding two-dimensional projective coordinates:

\[
\begin{bmatrix} X = x/z, & Y = y/z \end{bmatrix} \Rightarrow \begin{bmatrix} X' = x'/z', & Y' = y'/z' \end{bmatrix}
\]

Recall that the 3-D coordinates of a point on the object whose three-dimensional (3-D) spatial coordinates before and after motion are \((x, y, z)\) and \((x', y', z')\), respectively. Let \((X, Y) [(X', Y')]\) be its central projective coordinates before (after) motion onto the image plane, \(z = 1\), with the projective center at the origin \(O\). The following projective equations relate the 3-D spatial coordinates and their corresponding two-dimensional projective coordinates:

\[
\begin{bmatrix} T \times (X, Y, 1) \end{bmatrix} = \begin{bmatrix} T \times [R_0(X, Y, 1)] \end{bmatrix}
\]

and, after taking dot product of both sides of Eq. (3) with \((X', Y', 1)\),

\[
(X', Y', 1)(T \times R_0)(X, Y, 1) = 0,
\]

where \(T \times R_0 = [T \times r_1, T \times r_2, T \times r_3]\); \(r_1, r_2, r_3\) being the columns of \(R_0\). Let \(E = T \times R_0\). Then Eq. (4) states that, for any image-point correspondence pair \([X, Y, (X', Y')]\), the \(3 \times 3\) matrix \(E\) satisfies the following equation that is linear and homogeneous in the elements of \(E\):

\[
(X', Y', 1)E(X, Y, 1) = 0.
\]

Denote the set of all observed image-point correspondence pairs \((X_i, Y_i) \leftrightarrow (X'_i, Y'_i), i = 1, 2, \ldots, N\), by \(P\). Let

\[
A = \begin{bmatrix} X_1' & Y_1' & X_1 & Y_1 & X_1 & Y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ X_N' & Y_N' & X_N & Y_N & X_N & Y_N & 1 \end{bmatrix},
\]

\[
E = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix},
\]

\(h = (h_1, \ldots, h_9)^T\). (6)

Then it can be easily seen that the linear equations with \([X, Y, (X', Y')]\)
Y), (X', Y') ∈ P are equivalent to the following matrix linear equation for h:

\[ Ah = 0. \]  

(7)

Both Eqs. (5) and (7) will be called the two-view motion equations. Our approach is first to solve for the intermediate unknowns \( h_1 \) and then to obtain the motion parameters from them. Since any \( T \times R_o \) with \( T \times T_o = 0 \) satisfies both Eqs. (5) and (7) (the latter if \( T \times R_o \) is rearranged as \( h \)) and, moreover, since such a vector \( T \) that is collinear with \( T_o \) has one degree of freedom when \( T_o \neq 0 \) and three degrees of freedom when \( T_o = 0 \), the general solution of the two-view motion equation therefore has at least one degree of freedom when \( T_o \neq 0 \) and three degrees of freedom when \( T_o = 0 \). In other words, the coefficient matrix \( A \) in Eq. (7) has a rank no larger than 8 when \( T_o \neq 0 \) and no larger than 6 when \( T_o = 0 \). If the rank of \( A \) equals 8, then the translation \( T_o \) must be nonzero, and the general solution must have one degree of freedom and hence coincides with \( \alpha(T_o \times R_o) \), where \( \alpha \) is any real number. If the rank equals 6 and the translation \( T_o \) is zero, then the general solution must have three degrees of freedom and hence coincides with \( T_o \times R_o \), where \( T \) is any real vector.

B. Degeneracy and Surface Assumption

Definition

The two-view motion equation is called degenerate if the rank of \( A \) is less than 8 when \( T_o \neq 0 \) or less than 6 when \( T_o = 0 \). Thus, when the two-view motion equation is not degenerate, any nonzero solution \( E \) can be decomposed into \( T \times R_o \) with \( T \times T_o = 0 \). It is apparent that the degeneracy is equivalent to having a nonzero solution \( E \) such that \( E \) is not equal to \( T \times R_o \) for any \( T \) with \( T \times T_o = 0 \).

However, we have the following.

Lemma

A matrix \( E \) is equal to \( T \times R_o \) with \( T \times T_o = 0 \) if and only if \( R_o \times E + E \times R_o = 0 \) and \( T \times R_o = 0 \). (Proof of this lemma is given in Appendix A.)

From this lemma it is clear that the two-view motion equation is degenerate if and only if there is a nonzero solution \( E \) such that

\[ \| R_o \times E + E \times R_o \| + \| T \times E \| \neq 0. \]  

(8)

Since the rigid-body motion happens in the half-space \( z < 0 \), Eq. (5) is, as is easily seen, equivalent to

\[ (x', y', z')E(x, y, z)^t = 0, \]  

(9)

or, after substituting \( [R_o \times (x, y, z) + T] \) for \( (x', y', z') \) in Eq. (9),

\[ (x, y, z)(R_o \times E)(x, y, z)^t + T \times E(x, y, z)^t = 0. \]  

(10)

As a result, the two-view motion equation becomes degenerate if and only if Eq. (10) has a nonzero solution \( E \) such that inequality (8) holds when \( E \) is nonzero and \( (x, y, z) \) is visible before and after motion and produces the set of image-point correspondence pairs, \( P \). Letting \( U = R_o \times E \), then Eq. (10) for \( E \) is equivalent to the following equation for \( U \):

\[ (x, y, z)U(x, y, z)^t + T \times R_o U(x, y, z)^t = 0. \]  

(11)

Thus we obtain the following theorems.

Theorem 1

The two-view motion equation is not degenerate if and only if the surface assumption holds, that is, one cannot find a \( 3 \times 3 \) matrix \( U \) such that the group of surface points \( S \) are contained in the following quadratic surface:

\[ (x, y, z)U(x, y, z)^t + (T \times R_o U)(x, y, z)^t = 0, \]  

(12)

with \( \| U + U^t \| + T \times R_o U \| \neq 0 \).

Theorem 2

Under the surface assumption, the two-view motion equation has a rank 8 and a general solution \( \alpha T \times R_o \) (where \( \alpha \) is any real number) when \( T_o \neq 0 \) or a rank 6 and a general solution \( T \times R_o \) (where \( T \) is any real vector) when \( T_o = 0 \).

Because of theorem 2, at least six or eight image-point correspondence pairs are needed to ensure the surface assumption, depending on whether the translation \( T_o \) is zero or not. In practice, more pairs are preferable to increase the probability that the surface assumption will be satisfied and to smooth out noise effects. We would like to point out that our surface assumption is equivalent to the condition of Longuet-Higgins1 when \( T_o \neq 0 \). His condition does not include the case \( T_o = 0 \).

The surface assumption, as stated in theorem 1, has the following interpretation. To simplify the interpretation, we assume that the object is stationary and that the camera is moving. Let the origin of the cameras system be \( 0 \) and \( 0' \), respectively, before and after the motion. Then, for \( T_o \neq 0 \), the surface assumption holds if and only if the 3-D points corresponding to \( P \) do not lie on a quadratic surface passing through \( 0 \) and \( 0 \). For \( T_o = 0 \) (then \( 0 \) and \( 0' \) coincide), the surface assumption holds if and only if the 3-D points corresponding to \( P \) do not lie on a cone with its apex at \( 0 \).

C. Solving the Motion Equations

Now we come to the question of computing \( E \) or \( h \). There are a number of possibilities. We propose the following procedure. From eight or more point correspondences \( (X_i, Y_i) \leftrightarrow (X_i', Y_i') \), we form the positive semidefinite and symmetrical matrix \( W = A^tA \), where \( A \) is given by Eq. (6). Then we find \( h \) to minimize \( hh^tW \), using the constraint \( \| h \| = 1 \). The solution is the eigenvector of \( W \) associated with the smallest eigenvalue. The motivation for this method is as follows. In the absence of noise, we have

\[ Ah = 0. \]  

(7)

When noise is present in the image-point coordinates, Eq. (7) is no longer valid. A reasonable thing to do is to find a least-squares solution, i.e., to minimize

\[ \| Ah \|^2 = (Ah)^t(Ah) = h^tA^tAh = h^tWh. \]

Assume that the surface assumption holds. Then, if \( T_o \neq 0 \), the rank of \( W \) is 8 in the absence of noise. One and only one eigenvalue of \( W \) will be zero, and the corresponding eigenvector gives us the exact solution. In the presence of noise (which is assumed to be small), the smallest eigenvalue will be almost zero, and the corresponding eigenvector gives us a least-squares solution. When \( T_o = 0 \), the rank of \( W \) is 6 in the absence of noise. Three eigenvalues will be zero, and the corresponding eigenvector (after normalization to unit magnitude) will have two degrees of freedom. In the presence of (small) noise, three of the eigenvalues of \( W \) will be
Thus \( T \) can be determined up to a sign by solving the following equations:

\[
\|E_1\|^2 = t_2^2 + t_3^2, \\
\|E_2\|^2 = t_3^2 + t_1^2, \\
\|E_3\|^2 = t_1^2 + t_2^2, \\
(E_1, E_2) = t_1t_2, \\
(E_2, E_3) = t_2t_3, \\
(E_3, E_1) = t_3t_1
\]

\[(E_1, E_2)\) denotes the dot product of \(E_1\) and \(E_2\), \]

where, as is easily seen, Eqs. (18) are equivalent to Eqs. (22)–(24):

\[
\|E_3\|^2 + \|E_2\|^2 - \|E_1\|^2 = 2t_1^2, \quad (22) \\
\|E_3\|^2 + \|E_1\|^2 - \|E_2\|^2 = 2t_2^2, \quad (23) \\
\|E_1\|^2 + \|E_2\|^2 - \|E_3\|^2 = 2t_3^2. \quad (24)
\]

From noise considerations, we recommend a scheme to compute \( \pm T \) as follows:

Step 1. If \( |t_1| > |t_2|, |t_3| \) in Eqs. (22)–(24), then \( \pm T \) are determined by using Eqs. (22), (19), and (21). Stop.

Step 2. If \( |t_2| > |t_3| \) in Eqs. (23) and (24), then \( \pm T \) are determined by using Eqs. (23), (19), and (20). Stop.

Step 3. \( \pm T \) are determined by using Eqs. (24), (20), and (21). Stop.

B. Determining Rotation

Once \( \pm T \) are determined, \( R \) and \( R' \) could be computed by means of Eqs. (15)–(17) and (15')–(17'). In fact, a simple manipulation leads to

\[
E_1 	imes E_2 = t_2(t_1R_1 + t_2R_2 + t_3R_3), \\
E_2 	imes E_3 = t_1(t_1R_1 + t_2R_2 + t_3R_3), \\
E_3 	imes E_1 = t_3(t_1R_1 + t_2R_2 + t_3R_3),
\]

(25)

and hence

\[
t_1\|T\|^2R_1 = (E_2 	imes E_3) 	imes E_1 + t_1(E_2 	imes E_3), \quad (27) \\
t_2\|T\|^2R_1 = (E_3 	imes E_1) 	imes E_2 + t_2(E_3 	imes E_1), \quad (28) \\
t_3\|T\|^2R_1 = (E_1 	imes E_2) 	imes E_3 + t_3(E_1 	imes E_2), \quad (29)
\]

which, combined with Eqs. (15)–(17), determine \( R \). For instance, when \( |t_1| > |t_2|, |t_3| \), we use Eqs. (27), (17), and (16) to compute \( R_1, R_2, \) and \( R_3 \), respectively, and so on. Similarly, we could obtain

\[
-t_2\|T\|^2R_1 = (E_2 	imes E_3) 	imes E_1 - t_1(E_2 	imes E_3), \quad (27') \\
-t_3\|T\|^2R_1 = (E_3 	imes E_1) 	imes E_2 - t_2(E_3 	imes E_1), \quad (28') \\
-t_1\|T\|^2R_1 = (E_1 	imes E_2) 	imes E_3 - t_3(E_1 	imes E_2), \quad (29')
\]

which, combined with Eqs. (15')–(17'), determine \( R' \).

Thus we have outlined a direct procedure to compute \( T, R, R' \) from \( E \). In the next section we discuss how to determine the true rotation, the true translation direction, and the relative ranges of observed points from \( T, R, R' \). Here we would like to point out what happens with the decompositions when noise is present in the measurements of image-point coordinates. In general, an erroneous nonzero solution \( E \) does not admit any decompositions, as in Eq. (13). However, by using the above procedure, we still can compute a vector \( T \) and two matrices \( R \) and \( R' \). The triplet \( (T, R, R') \)
should approach the true triplet \((T, R, R')\) when the noise tends to zero. In other words, \((\hat{T}, \hat{R}, \hat{R}')\) should be closer to \((T, R, R')\) when the noise becomes smaller. The two matrices \(R\) and \(R'\) might not be orthonormal matrices of the first kind. However, algorithms exist for constructing two orthonormal matrices of the first kind, \(\hat{R}\) and \(\hat{R}'\), which are approximations of \(R\) and \(R'\) (and hence \(R\) and \(R'\), respectively; see, for example, Arun et al.).

4. Determining Three-Dimensional Motion Parameters and Surface Structure from T, R

A. Determining Rotation and Translation Direction

Under the surface assumption, any nonzero solution \(E\) admits two and only two decompositions, as in Eq. (13). The next task is to determine the true rotation from \(R\) and \(R'\), the true translation direction from \(\hat{T}\), and also the relative depths \(z/\|\hat{T}_a\|\) and \(z/\|\hat{T}_o\|\) when \(\hat{T}_o \neq 0\). What we really need is a criterion function \(L(\cdot, \cdot)\), where the first argument is a 3 \(\times\) 1 vector and the second a 3 \(\times\) 3 matrix such that \(L(T, R)\) equals zero if and only if \(R = R_o\) and \(T\) has the same direction as \(T_o\). Note that a zero vector has an indefinite direction; in other words, it has the same direction as any other vector. Thus such a function \(L\) should satisfy the following conditions:

\[
L(T, R) = 0, \quad L(-T, R) = 0, \quad L(\pm T, R') = 0.
\]

(30)

\[
\begin{cases}
L(T, R) = 0, & \text{if } R = R_o, \quad T_o \\
\|T\|/\|T_o\| = T_o/\|T_o\|, & \text{if } R = R_o, \quad T_o = 0.
\end{cases}
\]

(31)

To see how we should design such a criterion function, we return to the source information, the 3-D rigid-body motion equation (1). Suppose that \(R\) equals \(R_o\) and \(T\) has the same direction as \(T_o\). If \(T_o \neq 0\), then there must be a constant \(\alpha > 0\) such that \(T = \alpha T_o\). Thus it follows from Eqs. (1) and (2) that

\[
\alpha z(X', Y, 1) = \alpha z R(X, Y, 1) + T.
\]

(32)

hence

\[
\alpha z(X', Y, 1) = \alpha z R(X, Y, 1) + T \quad \text{and} \quad R \neq R_o, \quad T_o = 0.
\]

(33)

For abbreviation, we let

\[
v = (X, Y, 1),
\]

\[
v' = (X', Y', 1).
\]

(34)

Since \(\alpha > 0, z < 0, z' < 0\), from Eqs. (33) we obtain

\[
\alpha z' \|v' \times Rv\| = -\|T \times Rv\|,
\]

\[
\alpha z^2 \|v' \times Rv\| = -\|T \times v'\|.
\]

(35)

Then, multiplying both sides of Eq. (32) by \(\|v' \times Rv\|\), and substituting \(-\|T \times Rv\|\) for \(\alpha z' \|v' \times Rv\|\) and \(-\|T \times v'\|\) for \(\alpha z^2 \|v' \times Rv\|\) because of Eqs. (35), we have

\[
\|T \times Rv\|v' = -\|T \times v'\|Rv + \|v' \times Rv\|T,
\]

(36)

or, after rearrangement,

\[
\|T \times Rv\|v' - \|T \times Rv\|v + \|v' \times Rv\|T = 0.
\]

(37)

Denoting the left-hand side of Eq. (37) by \(H(v, v', T, R)\), we conclude that for each pair \((v, v') \in P\) the function \(H(v, v', T, R)\) equals zero whenever \(R\) equals \(R_o\) and \(T = \alpha T_o, \alpha > 0\) (assuming that \(T_o \neq 0\)). If \(T_o = 0\), then both \(H(v, v', T, R)\) and \(H(v, v', -T, R)\) equal zero whenever \(R\) equals \(R_o\), since in this case \(v'\) has the same direction as \(Rv\) and hence \(v' \times Rv = 0\) and \(\|T \times Rv\|v' = \|T \times v'\|Rv\). Thus, letting

\[
L(T, R) = \sum_{(v, v') \in P} \|H(v, v', T, R)\|/(\|v\| \cdot \|v'\| \cdot \|T\|),
\]

(38)

we have proved that \(L(T, R) = 0\) whenever \(R\) equals \(R_o\) and \(T\) has the same direction as \(T_o\). In what follows we verify that the function \(L(T, R)\) is just what we want. For this, the only thing that we need to verify is

\[
\begin{cases}
L(-T, R) > 0, & \text{if } R = R_o, \quad T_o = 0, \quad \text{and } T_o/\|T_o\| = T/\|T\| \quad \text{(39)}
\end{cases}
\]

and

\[
\begin{cases}
L(\pm T, R') > 0, & \text{if } R = R_o, \quad T_o = 0.
\end{cases}
\]

(40)

We would like to point out that the main purpose of the normalization and summation in Eq. (38) is to smooth out noise effects.

To prove expressions (39) and (40), we need to derive an explicit relation between \(R\) and \(R'\). From Eqs. (15)–(17) it follows that

\[
t_3(R_3 + R_o') = t_3(R_3 + R_o'),
\]

(41)

\[
t_1(R_3 + R_o') = t_1(R_3 + R_o'),
\]

(42)

\[
t_2(R_1 + R_o' + R_3) = t_2(R_1 + R_o' + R_3),
\]

(43)

and hence

\[
R + R' = T(R_1 + R_o')/t_1, \quad \text{when } t_1 \neq 0,
\]

(44)

\[
= T(R_2 + R_o' + R_3)/t_2, \quad \text{when } t_2 \neq 0,
\]

(45)

\[
= T(R_3 + R_o' + R_3)/t_3, \quad \text{when } t_3 \neq 0.
\]

(46)

In any case, there exists a row vector \(q\) such that

\[
R + R' = T \cdot q.
\]

(47)

Also, we need the following simple fact: Except for at most one pair \((v, v')\) in \(P\), the following inequalities hold:

\[
T \times v' \neq 0,
\]

(48)

\[
T \times Rv \neq 0.
\]

(49)

In fact, it is obvious that, except for at most one pair, inequality (48) holds. Then, when \(T_o \neq 0\), expressions (35) and (48) imply inequality (49); and when \(T_o = 0\), the nonzero vector \(Rv\) has the same direction as \(v'\) and hence inequality (49) also implies inequality (49).

Now we are ready to prove expressions (39) and (40). We need to prove that \(L(\pm T, R') > 0\) in general, and \(L(-T, R) > 0\) when \(T_o \neq 0\).

\[
L(\pm T, R') > 0.
\]

We need only to prove that, for a pair \((v, v')\) that satisfies inequalities (48) and (49), the function \(H(v, v', \pm T, R)\) is not equal to zero. As a matter of fact, an even stronger result exists: For any positive numbers, \(x'\) and \(\lambda\), the following inequality holds:

\[
\sum_{(v, v') \in P} \|H(v, v', T, R)\|/(\|v\| \cdot \|v'\| \cdot \|T\|) < \infty.
\]

(50)
\[ \lambda v' - \lambda Rv' \pm \|v' \times Rv'\|T \neq 0. \]  

Using relation (47) and the motion equation (1), we could rewrite \( \lambda v' - \lambda Rv' \) as

\[
\lambda v' - \lambda Rv' = \lambda v' - \lambda (-R + T \cdot q)v \\
= \lambda v' + \lambda Rv - \lambda T(qv) \\
= \left( \lambda + \frac{\lambda z'}{z} \right) v' - \frac{\lambda}{z} T_o - \lambda (qv)T. 
\]

Thus, the left-hand side of inequality (50) consists of two terms

\[
\left( \lambda + \frac{\lambda z'}{z} \right) v' \\
- \frac{\lambda}{z} T_o - \lambda (qv)T \pm \|v' \times Rv'\|T. 
\]

The coefficient of \( v' \) in the first term (52) is positive. The second term (53) is collinear with \( T \). And the two vectors, \( v' \) and \( T \), are not collinear with each other because of inequality (48). Thus, the sum, the left-hand side of inequality (50), cannot be zero. This completes the proof of \( L(\pm T, R') > 0 \).

\( L(-T, R) > 0 \) when \( T_o \neq 0 \): This is true since, except for at most one pair \((v, v')\), \( v' \times Rv \) does not equal zero when \( T_o \neq 0 \) because of expressions (35) and (48), and hence \( H(v, v', -T, R) \) cannot be zero:

\[
H(v, v', -T, R) = H(v, v', T, R) - 2\|v' \times Rv\|T \\
= -2\|v' \times Rv\|T \neq 0. 
\]

So, finally, we have proved the theorem given below.

**Theorem 3**

Assume that the surface assumption holds and that \( E = T \times R = (-T) \times R' \) is a nonzero solution of the two-view motion equation. Then \( R = R_o \) and \( T \) has the same direction as \( T_o \) if and only if

\[ L(T, R) = 0. \]  

**B. Determining Relative Depths**

Now it is easy to prove the following.

**Theorem 4**

Assume that \( R \) equals \( R_o \) and \( T \) has the same direction as \( T_o \). Then, when \( T_o \neq 0 \), the relative depths are given by

\[
z/\|T_o\| = -\frac{\|T \times v'\|}{\|T\| \cdot \|v' \times R_o v\|} \\
z'/\|T_o\| = -\frac{\|T \times R_o v\|}{\|T\| \cdot \|v' \times R_o v\|} 
\]

**Proof.** From the assumptions, Eqs. (35) (with \( \alpha = \|T\|/\|T_o\| \)) hold. Thus Eqs. (55) immediately follow, where the minuses are due to \( z < 0, z' < 0 \).

It is easy to argue that, except at most one point correspondence pair, \( T \times v' \) is nonzero. Thus, except at most one pair, \( v' \times R_o v \) is nonzero by Eqs. (35). This indicates that the division in Eqs. (55) is not a problem.

**Q.E.D.** Theorems 3 and 4 indicate that the rotation, the translation direction, and the relative depth map can all be determined under the surface assumption without knowing the mode of the motion, i.e., irrespective of whether the translation is zero.

**C. Noise Effects**

For convenience, we could modify theorem 3 as follows: \( R \) equals \( R_o \) and \( T \) has the same direction as \( T_o \) if and only if

\[
\min[L(T, R), L(-T, R)] < \min[L(T, R'), L(-T, R')]. 
\]

where relation (56) is used to determine the true rotation and after that relation (57) is used to determine the true translation direction. The equal sign in relation (57) is possible only when \( T_o = 0 \).

When noise appears in the measurements, the triplet \((T, R, R')\) cannot be accurately computed. However, if the noise is small, the computed triplet \((\hat{T}, \hat{R}, \hat{R'})\) will be close to \((T, R, R')\), and, as is easily seen, \( L(\pm \hat{T}, \hat{R}) \) and \( L(\pm \hat{T}, R) \) will also be close to \( L(\pm T, R) \) and \( L(\pm T, R') \), respectively. Therefore relation (56) will imply that

\[
\min[L(\hat{T}, \hat{R}), L(-\hat{T}, \hat{R})] < \min[L(T, R'), L(-T, R')]. 
\]

and relation (57) when \( T_o \neq 0 \) will imply that

\[ L(\hat{T}, \hat{R}) < L(-\hat{T}, \hat{R}). \]

As a result, relation (56') and the following relation (57') should give correct approximations of the rotation and the translation direction:

\[ L(\hat{T}, \hat{R}) \leq L(-\hat{T}, \hat{R}). \]

**5. SUMMARY OF THE ALGORITHM**

Now we are ready to give the following unified algorithm that does not require the mode of motion to be known.

**Step 1**

Find \( h \) to minimize \( h^TWh \) under the constraint \( \|h\| = 1 \). (If the solution is not unique, pick any solution.)

**Step 2**

Let

\[ E_1 = (h_1, h_2, h_3), \]

\[ E_2 = (h_4, h_5, h_6), \]

\[ E_3 = [E_1 \ E_2 \ E_3]. \]

**Step 3**

\[ a = (\|E_3\|^2 + \|E_2\|^2 - \|E_1\|^2)/2, \]

\[ b = (\|E_3\|^2 + \|E_1\|^2 - \|E_2\|^2)/2, \]

\[ c = (\|E_1\|^2 + \|E_2\|^2 - \|E_3\|^2)/2. \]
Step 4
If \((a \geq b, c)\) then let
\[
\begin{bmatrix}
  t_1 \\
  t_2 \\
  t_3
\end{bmatrix} = \begin{bmatrix}
 \sqrt{a} \\
 -(E_1, E_2)/\sqrt{a} \\
 -(E_1, E_3)/\sqrt{a}
\end{bmatrix},
\]
\[
R_1 = \frac{[(E_2 \times E_3) \times E_1 + t_1(E_2 \times E_3)\|T\|^2]}{(t_1\|T\|^2)},
R'_1 = \frac{[(E_2 \times E_3) \times E_1 - t_1(E_2 \times E_3)\|T\|^2]}{(-t_1\|T\|^2)},
R_2 = \frac{(E_3 - t_2R_1)/t_1}{(-t_1)},
R_3 = \frac{(-E_2 + t_3R_2)/t_3}{(-t_1)}
\]
and GO TO Step 7.

Step 5
If \((b \geq c)\), then let
\[
\begin{bmatrix}
  t_1 \\
  t_2 \\
  t_3
\end{bmatrix} = \begin{bmatrix}
 \sqrt{b} \\
 -(E_1, E_2)/\sqrt{b} \\
 -(E_1, E_3)/\sqrt{b}
\end{bmatrix},
\]
\[
R_2 = \frac{[(E_2 \times E_3) \times E_2 + t_2(E_2 \times E_3)\|T\|^2]}{(t_2\|T\|^2)},
R'_2 = \frac{[(E_2 \times E_3) \times E_2 - t_2(E_2 \times E_3)\|T\|^2]}{(-t_2\|T\|^2)},
R_3 = \frac{(E_1 - t_3R'_2)/t_2}{(-t_2)},
R'_3 = \frac{(-E_2 - t_3R'_2)/t_3}{(-t_2)}
\]
and GO TO Step 7.

Step 6
Let
\[
\begin{bmatrix}
  t_1 \\
  t_2 \\
  t_3
\end{bmatrix} = \begin{bmatrix}
 -(E_2, E_3)/\sqrt{c} \\
 -(E_3, E_2)/\sqrt{c} \\
 \sqrt{c}
\end{bmatrix},
\]
\[
R_3 = \frac{[(E_1 \times E_2) \times E_3 + t_3(E_1 \times E_2)\|T\|^2]}{(t_3\|T\|^2)},
R'_3 = \frac{[(E_1 \times E_2) \times E_3 - t_3(E_1 \times E_2)\|T\|^2]}{(-t_3\|T\|^2)},
R_1 = \frac{(E_2 + t_3R_3)/t_3}{(-t_3)},
R'_1 = \frac{(E_2 - t_3R'_3)/t_3}{(-t_3)},
R_2 = \frac{(-E_1 + t_3R_3)/t_3}{(-t_3)},
R'_2 = \frac{(-E_1 - t_3R'_3)/(-t_3)}{-t_3}
\]

Step 7
Let
\[
R = \begin{bmatrix}
 R_1 \\
 R_2 \\
 R_3
\end{bmatrix},
\]

Step 8
If
\[
\min[L(T, R), L(-T, R)] < \min[L(T, R'), L(-T, R')],
\]
then
\[
R_o = R.
\]
Otherwise
\[
R_o = R'.
\]

Step 9
If
\[
L(T, R_o) \leq L(-T, R_o),
\]
then \(T_o\) has the same direction as \(T\). Otherwise, \(T_o\) has the same direction as \((-T)\).

Step 10
When \(T_o \neq 0\), the relative depths are given by
\[
z = \frac{\|Txv\|}{\|T\|} \frac{\|Tv\|}{\|Tv\|^2}, \quad z' = \frac{\|Txv\|}{\|T\|} \frac{\|Tv\|}{\|Tv\|^2}.
\]

Step 11.
STOP.

Simulation 1
\[
T_o = (0, 0, 0), \quad R_o = \begin{bmatrix}
  1/\sqrt{2} & 1/\sqrt{2} & 0 \\
 -1/\sqrt{2} & 1/\sqrt{2} & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]
Six points in image plane \(z = 1\) before motion:
\[
(0.63, -0.93),
(2.09, 0.10),
(0.53, 1.43),
(1.85, 1.83),
(1.29, 0.41),
(-1.32, -0.12).
\]
Six points in image plane \(z = 1\) after motion:
\[
(-0.21, -1.10),
(1.54, -1.41),
(1.39, 0.63),
(2.60, -0.01),
(1.20, -0.62),
(-1.01, 0.85),
\]
Computed $E, R, R', T$:

$$E = \begin{bmatrix} 0.27 & -0.27 & -0.02 \\ 0.27 & 0.27 & -0.59 \\ -0.41 & 0.43 & -0.00 \end{bmatrix},$$

$$R = \begin{bmatrix} 0.71 & 0.71 & 0.00 \\ -0.71 & 0.71 & 0.00 \\ 0.00 & -0.00 & 1.00 \end{bmatrix},$$

$$R' = \begin{bmatrix} 0.32 & 0.25 & 0.91 \\ 0.67 & -0.74 & -0.03 \\ 0.67 & 0.63 & -0.40 \end{bmatrix},$$

$$T = \begin{bmatrix} 0.59 \\ -0.02 \\ 0.39 \end{bmatrix}.$$

$L(T, R) = 0.00,$

$L(-T, R) = 0.00.$

$L(T, R') = 6.95.$

$L(-T, R') = 6.95.$

$\min[L(T, R), L(-T, R)] = 0.00.$

$\min[L(T, R'), L(-T, R')] = 6.95.$

Thus the algorithm gives the correct rotation $R_0 = R.$

Simulation 2

$T_o = (0, 0, -1)^t,$ $R_o = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Eight points in image plane $z = 1$ before motion:

$(-0.04, 0.96), (-0.09, -1.22), (-0.67, 0.91), (1.17, 1.29), (1.10, 0.65), (-0.13, -0.98), (-1.13, -1.19), (1.03, -0.37).$

Eight points in image plane $z = 1$ after motion:

$(0.41, 0.44), (-0.60, -0.52), (0.10, 0.67), (1.07, 0.06), (0.62, -0.16), (-0.45, -0.35), (-0.89, -0.02), (0.29, -0.62).$

APPENDIX A

Lemma

$E = T \times R_o$ with $T \times T_o = 0$ if and only if $R_o' E + E' R_o = 0$ and $T_o' E = 0.$

Proof

Only-If Part

Assume that $E = T \times R_o$ with $T \times T_o = 0$. Then it follows that

$E = \begin{bmatrix} T \times r_1, T \times r_2, T \times r_3 \end{bmatrix},$

$R_o' E = \begin{bmatrix} (r_o, T \times r_j) \end{bmatrix},$

$E' R_o = \begin{bmatrix} (T \times r_j, r_j) \end{bmatrix},$

$T_o' E = \begin{bmatrix} (T_o, T \times r_j), (T_o, T \times r_2), (T_o, T \times r_3) \end{bmatrix},$

where $R_o = [r_1, r_2, r_3].$ And hence we obtain

$R_o' E + E' R_o = 0 \quad \text{by} \quad (r_o, T \times r_j) + (T \times r_1, r_j) = 0,$

$T_o' E = 0 \quad \text{by} \quad (T_o, T \times r_j) = 0.$

If Part

Assume that $R_o' E + E' R_o = 0$ and $T_o' E = 0$. Let $G = E R_o'$. Then it is easy to see that

$G^t = R_o E^t = R_o (-R_o' E R_o' E) = -E R_o' E,$

which indicates that $G$ is a skew-symmetrical matrix. Thus there is a $T$ (see Yen and Huang) so that
\[ E = GR_o = T \times R_o \]

and hence

\[ (T_o \times T)^t = T_o^t G = T_o^t E R_o^t = 0, \]

which completes the "if" part. Q.E.D.

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