proving the original ESPRIT method when the sensors are perturbed. All the advantages are obtained with a reasonable increase of computation load. However, the array manifold must be known, in contrast with the original ESPRIT.

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Maximum Entropy Image Reconstruction

Xinhua Zhuang, Robert M. Haralick, and Yunxin Zhao

***Abstract*—This correspondence justifies the maximum entropy image reconstruction (MEIR) formulation proposed by Zhuang *et al.* (1987), solves an open problem of whether or not there exists a solution to the MEIR problem, proves the fast convergence of the MEIR algorithm proposed by Zhuang *et al.*, and, finally, shows that the system of differential equations which is the basis of the MEIR algorithm is of the Lyapunov type.**

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X. Zhuang is with the Department of Electrical and Computer Engineering, University of Missouri, Columbia, MO 65211.

R. M. Haralick is with the Department of Electrical Engineering, University of Washington, Seattle, WA 98195.

Y. Zhao is with Speech Technologies Laboratory, Panasonic Technologies Inc., Santa Barbara, CA 93105.

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I. Introduction

This correspondence represents a continuation of the work reported in an earlier published paper by Zhuang *et al.* [7]. This correspondence discusses the following:

1) it justifies the proposed maximum entropy image reconstruction (MEIR) formulation;
2) proves the existence of the solution to the MEIR problem, an open problem (see [6]);
3) shows the fast convergence of the proposed MEIR algorithm;
4) demonstrates the MEIR algorithm is governed by a Lyapunov system, whose energy function measures the degree of constraint satisfaction.

This correspondence is organized as follows. Section II covers point 1) above. Section III covers 2)–4). The final section presents further research directions and a conclusion.

II. Justification of MEIR Formulation

Let the required reconstructed image have positive pixel values $f_1, \cdots, f_n$ which are to be determined, and on which the entropy

$$H(p_1, \cdots, p_n) = -\sum_i p_i \log p_i \tag{1}$$

is defined, where

$$p_i = \frac{f_i}{\sum_k f_k}, \qquad i = 1, \cdots, n. \tag{2}$$

Let observed image data be given by

$$d_j = \sum_i A_{ji} f_i + e_j, \qquad i = 1, \cdots, m \tag{3}$$

where e_j's are independent, zero mean, σ_j^2 variance noise terms. We assume σ_j^2 is known and define a constraint satisfaction function as follows:

$$Q(f_1, \cdots, f_n) = \frac{1}{2m} \sum_{j=1}^{m} \frac{\left(\sum_{i=1}^{n} A_{ji} f_i - d_j\right)^2}{\sigma_j^2}. \tag{4}$$

Typical least squares approaches would like to determine those values $f_1, \cdots, f_n$ which minimize $Q(f_1, \cdots, f_n)$. Rather than this, we seek those $f_1, \cdots, f_n$ which maximize the entropy $H(p_1, \cdots, p_n)$ subject to the constraint

$$Q(f_1, \cdots, f_n) = \tfrac{1}{2} \tag{5}$$

which comes about from the central limit theorem [1], that is, with probability one

$$Q(f_1, \cdots, f_n) \to \tfrac{1}{2}(m \to \infty). \tag{6}$$

Thus, provided m is large, we would expect the true value of $f_1, \cdots, f_n$ to satisfy (5). The condition (5) determines the set of feasible images each of which satisfies the given statistical test for consistency with the actual image data $\{d_1, \cdots, d_m\}$. In summary, the MEIR problem can be formulated as follows:

$$\begin{aligned} &\max_{f_1,\cdots,f_n} H(p_1, \cdots, p_n) \\ &\text{subject to} \quad Q(f_1, \cdots, f_n) = \tfrac{1}{2} \quad \text{and} \sum_i p_i = 1. \end{aligned} \tag{7}$$

Because of the linear relation between $H(p_1, \cdots, p_n)$ and $H(f_1, \cdots, f_n)$, i.e.,

$$H(p_1, \cdots, p_n) = \frac{H(f_1, \cdots, f_n)}{\sum_i f_i} - \log \sum_i f_i,$$

it is easy to verify that the formulation (7) is equivalent to

$$\begin{cases} \max_{\beta,\alpha,f} \left[\dfrac{H(f)}{\alpha} - \log \alpha + \beta\left(\sum_i f_i - \alpha\right)\right] \\ \text{subject to} \quad Q(f) = \dfrac{1}{2} \end{cases} \tag{8}$$

where $f = (f_1, \cdots, f_n)'$ and α, β are Lagrange multipliers. The constrained maximization (8) can be further rearranged as follows:

$$\begin{cases} \max_{\beta,\alpha} \left\{\dfrac{1}{\alpha}\left[\max_f \left(H(f) + \alpha\beta \sum_i f_i\right)\right] - \log \alpha - \alpha\beta\right\} \\ \text{subject to} \quad Q(f) = \dfrac{1}{2}. \end{cases} \tag{9}$$

As clearly seen, the main part for (9) is the following constrained maximization:

$$\begin{cases} \max_f \left(H(f) + \mu \sum_i f_i\right) \\ \text{subject to } Q(f) = \tfrac{1}{2} \end{cases} \tag{10}$$

where $\mu = \alpha\beta$. Finally, the formulation (10) is equivalent to the following constrained maximization:

$$\begin{cases} \max_f J(f; \mu, t) = H(f) + \mu \sum_i f_i - tQ(f) \\ \text{subject to} \quad Q(f) = \tfrac{1}{2} \end{cases} \tag{11}$$

where t is another Lagrange multiplier.

By now, the MEIR formulation proposed by Zhuang *et al.* [7], which was based on some heuristic consideration, has been justified since it is exactly the same as (11), the latter is formally derived as the main part of the original formulation (7). As argued in Zhuang *et al.*, usually we are satisfied with the solution to (11), where μ can be arbitrary, if not, there exists an efficient way to adjust μ to get a more satisfactory solution.

In the next section, we show the existence and uniqueness of the solution to the MEIR problem formulated as (11) and prove the fast convergence of the MEIR algorithm for solving (11).

III. Existence, Uniqueness, and Fast Convergence

It was proved by Zhuang *et al.* [7], that for each fixed μ and each fixed $t \geq 0$, the function $J(f; \mu, t)$ reaches its unique maximal point denoted as $f(t; \mu)$ internally, i.e., $f_i(t; \mu) > 0$, $i = 1, \cdots, n$, furthermore $f(t; \mu)$ $(t \geq 0)$ coincides with the solution curve determined by the stationary point equation, i.e., $\nabla J(f; \mu, t) = 0$ $(t \geq 0)$, or, equivalently, determined by the following initial value problem of differential equations:

$$\begin{cases} (\nabla^2 J) \dfrac{df}{dt} = \nabla Q, \qquad t > 0 \\ f(0; \mu) = \exp(\mu - 1)(1, \cdots, 1)'_{1\times n} \end{cases} \tag{12}$$

where the initial value, i.e., $f(0; \mu) = \exp(\mu - 1)(1, \cdots, 1)'_{1\times n}$, solves $\nabla J(f; \mu, 0) = 0$. The initial value problem comprises the basis of our MEIR algorithm.

In order to prove the existence and the uniqueness of the solution to the MEIR problem formulated as (11), we will show, along the solution curve $f(t; \mu)$ $(t \geq 0)$ defined by (12), that the constraint satisfaction function $Q(f(t; \mu))$ approaches $\frac{1}{2}$ as $t \to \infty$ or at a unique finite time t^*.

Assume

$$\text{MIN} = \min_{f_1 \geq 0, \cdots, f_n \geq 0} Q(f) \leq \tfrac{1}{2}. \tag{13}$$

We can prove even more, that is,

$$Q(f(t; \mu)) - \text{MIN} \leq O\left(\frac{1}{t}\right). \tag{14}$$

Let f^0 be an arbitrary point in S which is defined as follows:

$$S = \{f\colon Q(f) = \text{MIN, with each } f_i \geq 0,\ i = 1, \cdots, n\}. \tag{15}$$

By the definition of $f(t; \mu)$ it follows:

$$\begin{aligned} J(f(t; \mu); \mu, t) &\geq J(f^0; \mu, t) \\ &= -\sum_i f_i^0 \log f_i^0 + \mu \sum_i f_i^0 - tQ(f^0) \\ &= \text{const.} - t\,\text{MIN}. \end{aligned} \tag{16}$$

Thus

$$\begin{aligned} -\sum_i f_i(t; \mu) \log f_i(t; \mu) &+ \mu \sum_i f_i(t; \mu) \\ &\geq t[Q(f(t; \mu)) - \text{MIN}] + \text{const.} \\ &\geq \text{const.} \\ &> -\infty. \end{aligned} \tag{17}$$

Since the left side of (17) tends to $-\infty$ as some $f_i \to \infty$, we conclude that there must exist $0 < B < \infty$ so that for each i and $t \geq 0$

$$0 < f_i(t; \mu) \leq B, \qquad i = 1, \cdots, n. \tag{18}$$

As a result, we get

$$\begin{aligned} Q(f(t; \mu)) - \text{MIN} &\leq \frac{1}{t}\left\{-\sum f_i(t; \mu) \log f_i(t; \mu) + \mu \sum_i f_i(t; \mu) - \text{const.}\right\} \\ &\leq \frac{1}{t} O(1) \end{aligned}$$

which straightforwardly leads to (14).

If $\text{MIN} < \frac{1}{2}$, then there exists a unique t^* such that

$$Q(f(t^*; \mu)) = \tfrac{1}{2}. \tag{19}$$

It is apparent that $f(t^*; \mu)$ maximizes $J(f; t^*, \mu)$, satisfies (19), and hence uniquely solves the MEIR problem formulated as (11).

If $\text{MIN} = \frac{1}{2}$, we can further prove that $\lim_{t\to\infty} f(t; \mu)$ exists and uniquely solves (11).

Finally, we notice that

$$\nabla^2 J(f(t; \mu); \mu, t) < 0$$

$$\frac{dQ(f(t; \mu))}{dt} = (\nabla Q)'(\nabla^2 J)^{-1}(\nabla Q) \leq 0.$$

Thus, along the solution curve $f(t; \mu)$ ($t \geq 0$), the constraint satisfaction function $Q(f(t; \mu))$ will monotonically decrease to MIN as $t \rightarrow \infty$ and hence qualify to be the energy function of the system (12). This concludes that the system (12), i.e., the basis of MEIR algorithm, is of the Lyapunov type.

IV. Further Research and Conclusions

Some subjects of further research are as follows.

1) Assume e_j's are i.i.d., $e_j \sim \mathfrak{N}(0, \sigma^2)$ with unknown σ^2. How should the MEIR problem be formulated?

2) Assume e_j's are i.i.d.

$$e_j \sim (1 - \epsilon)\mathfrak{N}(0, \sigma^2) + \epsilon h(e_j)$$

where σ, ϵ, $h(\cdot)$ unknown. How should the MEIR problem be formulated?

In this correspondence, we proved the existence and uniqueness of the solution to the MEIR problem and derived the fast convergence of the previously proposed MEIR algorithm. For a detailed exposition of the maximum entropy principle in image recovery and related references, please refer to [5].

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