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THE CHARACTERIZATION OF BINARY RELATION HOMOMORPHISMS

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A characterization theorem is derived for homomorphisms of one binary relation to another. The characterization theorem states that any homomorphism from one relation to another can be represented as the intersection of some basis relations. Furthermore, appropriate intersections of the basis relations define a relation homomorphism.

The characterization theorem leads to an efficient algorithm for determining all the homomorphisms from one relation to another: first find the basis relations, and then use the basis relations to generate all single-valued relations which then will be all the homomorphisms from one relation to the other.

Specialization of the procedure easily determines whether two relations or digraphs are isomorphic.

INDEX TERMS Relation homomorphism, relation, relation isomorphism, digraph, graph, digraph homomorphism, digraph isomorphism, graph homomorphism, graph isomorphism, relation composition.

1 INTRODUCTION

Characterizing binary relation homomorphisms is an important task in certain kinds of systems, chemical compound matching, information retrieval, artificial intelligence, linguistic and social network problems. For example, the interconnection of subsystems of a large system can be thought of as a binary relation and one interesting system problem is to determine whether two systems are isomorphic. The structure these homomorphism problems usually present themselves in is the graph structure. However, for mathematical simplicity in this paper, we use the binary relation structure. If A is a set, a binary relation R on the set A is defined as a subset of $A \times A$; $R \subseteq A \times A$. The digraph of a binary relation R on a set A is a graph whose nodes correspond to the elements in A and whose directed links correspond to the ordered pairs in R.

Research (Unger,[5] Corneil and Gotlieb[2]) on graph isomorphisms often lead to algorithms which determine a partition of the nodes and successively refine the partition using necessary conditions until a stable partition is reached. Berztiss[1] gives a node partitioning backtrack procedure for finding whether two digraphs are isomorphic.

Rather than using the necessary conditions to successively refine a partition on the set A, we suggest as Ullmann[4] using the necessary conditions to eliminate matching certain pairs of one relation to certain pairs of the other. This, in effect, partitions the arcs of the digraph or the pairs in the binary relation R. The partition on R determines a partition on A. What is interesting about this approach is that using a winnowing process "basic relations" whose intersections are the homomorphisms from one relation to another can be quickly computed.

To begin we need some definitions

DEFINITION Let $R \subseteq A \times A$, and $H \subseteq A \times B$. Define the composition of R with H by

$$R \circ H = \{(b, b') \varepsilon B \times B \mid \text{for some } (a, a') \varepsilon R, (a, b) \varepsilon H \text{ and } (a', b') \varepsilon H\}$$

Note that this definition of composition is different from the usual definition of binary relation composition. We have made this change of definition in order to facilitate the picture that a pair (a, a') in relation R can be mapped to the pair (b, b') only under the condition that the relation H associates b with a and b' with a'. In other words, pairs get mapped to pairs under a mapping which makes the same association for each component of the pair.

Let $R \subseteq A \times A$ and $S \subseteq B \times B$. For a relation $H \subseteq A \times B$ to be a homomorphism of R to S we will insist that H be capable of mapping each pair in R to some pair in S. We allow some pairs in S to be the image of no pair in R. The idea of this kind of homomorphism is illustrated by the kind of square diagram shown in Figure 1.

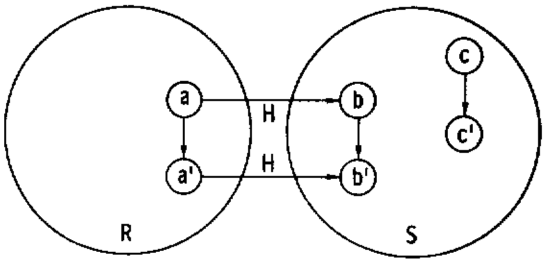

FIGURE 1 illustrates how a homomorphism maps each pair in R to some pair in S but there may be some pairs in S with no pre-images in R.

DEFINITION Let $R \subseteq A \times A$, $S \subseteq B \times B$, and $H \subseteq A \times B$.

H is a homomorphism of R into S if and only if

1) H is defined everywhere on A (for every $a \varepsilon A$, there exists a $b \varepsilon B$ such that $(a, b) \varepsilon H$)

2) H is single-valued on $B((a, b) \varepsilon H$ and $(a, b') \varepsilon H$ imply $b = b'$)

3) $R \circ H \subseteq S$.

The relation $R \circ H$ is called the homomorphic image of R under H.

This definition of homomorphism is the same as Harary.[3] The specific problem that we are interested in is: given relations R and S, determine all the homomorphisms of R into S.

2 THE WINNOWING PROCESS

Suppose that the relation $H \subseteq A \times B$ is a homomorphism of R into S, then what can we find out about the relation among H, R, and S? Certainly if H is a homomorphism from R to S and if H associates b with a, and if R associates a' with a, then all the elements which H associates with a' must be a subset of all the elements which S associates with b. This is illustrated in Figure 2. Likewise if H is a homomorphism from R to S and if H associates b with a, and if R associates a with a', then all the elements which H associates with a' must be a subset of all the elements which S^{-1} associates with b. This is illustrated in Figure 3.

As a direct consequence of this fact, if H is a homomorphism of R to S and a belongs to the domain of R, then whatever H associates with a must be contained in the domain of S. Likewise if H is a homomorphism of R to S and a belongs to the range of R, then whatever H associates with a must be contained in the range of S.

Figure 4 shows two simple binary relations and their corresponding digraphs which we will use as an example for the rest of the paper. For these relations we have Dom $S = \{a, b, c, d\}$ and Range $S = \{b, c, d, e\}$. Since 1 is in the domain of R, any homomorphism can only associate 1 with some of the elements in $\{a, b, c, d\}$. Since 2 or 5 is in the range of R, any homomorphism can only associate 2 or 5 with some of the elements in $\{b, c, d, e\}$. Since 3 and 4 are both in the range and domain of R, any homomorphism can only associate 3 or 4 with some of the elements in $\{a, b, c, d\} \cap \{b, c, d, e\} = \{b, c, d\}$. Thus any homomorphism H of R to S must be a subset of the relation T shown in Figure 5.

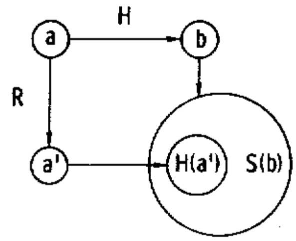

FIGURE 2 illustrates how functions which are homomorphisms are constrained.

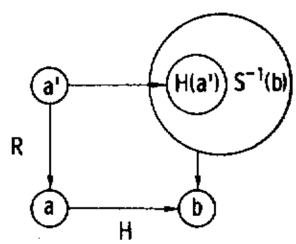

FIGURE 3 illustrates how functions which are homomorphisms are constrained.

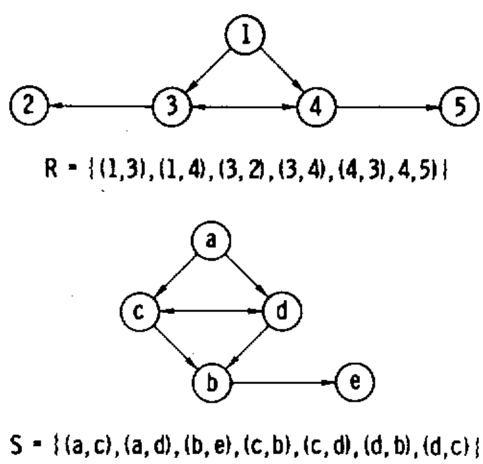

FIGURE 4 illustrates the diagraphs for the example relations R and S.

T	
1	a,b,c,d
2	b,c,d,e
3	b,c,d
4	b,c,d
5	b,c,d,e

FIGURE 5 shows the consequences of tracing the constraints. If H is a homomorphism of relation R to relation S (shown in Figure 4), then H must be a subset of the relation T.

This idea of determining more restricted relations which must contain any homomorphism can be generalized. Suppose it is known that T is a relation which contains a homomorphism H but the homomorphism H is not known. The relation T can be used to match pairs from the relation R to the relation S: the pair (a_1, a_2) of R is matched to the pair (b_1, b_2) of S only if $(a_1, b_1)\varepsilon T$ and $(a_2, b_2)\varepsilon T$. This set of matched pairs can then be reduced for if it is the case that a_1 is associated with b_1 by T, then unless each pair of R having a component of value a_1 can be matched by T with one pair of S whose corresponding component has value b_1, the homomorphism T contains will not match a_1 to b_1. Hence, if the pair (a_1, b_1) is in T and there is some pair of R having a component of value a_1 and this pair of R cannot be matched by T to some pair in S having a corresponding component value of b_1, then the pair (a_1, b_1) can be removed from T and T will still contain all the homomorphisms T originally contained.

We call this process of iteratively taking inconsistent pairs out of T the winnowing process and it bears a close relationship to the Waltz filtering process (Waltz 1972, 1975).[7] Proposition 1 gives a formal statement of the winnowing process and proves that after winnowing the inconsistent associations out of a given relation, the new relation will contain all the homomorphisms it originally contained. As an immediate consequence of Proposition 1, its corollary states that homomorphisms are fixed points of the winnowing process. This suggests that the winnowing process is a natural one to consider for determining homomorphisms.

Before stating and proving Proposition 1, its corollary, or other propositions, we will need some convenient notational conventions. Let $R\subseteq A_1 \times A_2$. We will define the following sets related to R:

$$\Delta_n R = \{a_n \varepsilon A_n \mid \text{for some } (a_1, a_2)\varepsilon A_1 \times A_2,\ (a_1, a_2)\varepsilon R\},\ n=1 \text{ or } 2;$$

$$R_n(a) = \{(a_1, a_2)\varepsilon R \mid a_n = a\},\ n=1 \text{ or } 2.$$

PROPOSITION 11 *The Winnowing Process.* Let $R\subseteq A\times A$, $S\subseteq B\times B$, $H\subseteq T\subseteq A\times B$.

Define one iteration of the winnowing process by

$$G=\{(a_1, a_2, b_1, b_2)\varepsilon R\times S \mid (a_1, b_1)\varepsilon T \text{ and } (a_2, b_2)\varepsilon T\}$$

$$Q=\left\{(a_1, b)\varepsilon T \,\middle|\, b\varepsilon \bigcap_{n=1}^{2} \bigcap_{(a_1,a_2)\varepsilon R_n(a)} \Delta_n G(a_1, a_2)\right\}$$

If H is defined everywhere and $R\circ H\subseteq S$, then $H\subseteq Q\subseteq T$.

Proof Let $(a, b)\varepsilon H$. To show $(a, b)\varepsilon Q$, we need only show

$$b\varepsilon \bigcap_{n=1}^{2} \bigcap_{(a_1,a_2)\varepsilon R_n(a)} \Delta_n G(a_1, a_2).$$

If for any n, $R_n(a)=\emptyset$, we have

$$\bigcap_{(a_1,a_2)\varepsilon R_n(a)} \Delta_n G(a_1, a_2) = B.$$

Since $H\subseteq T$, $(a_1, b)\varepsilon H$ implies $(a_1, b)\varepsilon T$ so that $(a_1, b)\varepsilon Q$ and this case causes no problem.

Let n be any index such that $R_n(a)\neq\emptyset$. We will show

$$b\varepsilon \bigcap_{(a_1,a_2)\varepsilon R_n(a)} \Delta_n G(a_1,a_2).$$

The argument for $n=1$ or $n=2$ is similar so without loss of generality, suppose $n=1$.

Let $(a_1, a_2)\varepsilon R_1(a)$. By definition of $R_1(a)$, we must have $a_1=a$; now for convenience we let $b_1=b$ so that $(a_1, b_1)=(a_1, b)\varepsilon H$. Since H is defined everywhere, there exists b_2 such that $(a_2, b_2)\varepsilon H$. Now, $(a_1, a_2)\varepsilon R$, $(a_1, b_1)\varepsilon H$, and $(a_2, b_2)\varepsilon H$ imply $(b_1, b_2)\varepsilon R\circ H$. But by supposition, $R\circ H\subseteq S$ so that $(b_1, b_2)\varepsilon R\circ H$ implies $(b_1, b_2)\varepsilon S$. And $(a_1, a_2)\varepsilon R$, $(b_1, b_2)\varepsilon S$, $(a_1, b_1)\varepsilon H$, $(a_2, b_2)\varepsilon H$, and $H\subseteq T$ imply $(a_1, a_2, b_1, b_2)\varepsilon G$. Hence $b=b_1\varepsilon\Delta_1 G(a_1, a_2)$ and $(a, b)\varepsilon Q$.

COROLLARY (*Homomorphisms are fixed points of the winnowing process*).

Suppose $R\circ H\subseteq S$ and H is defined everywhere. Define

$$G=\{(a_1, a_2, b_1, b_2)\varepsilon R\times S \mid (a_1, b_1)\varepsilon H \text{ and } (a_2, b_2)\varepsilon H\}$$

$$Q=\left\{(a_1, b)\varepsilon H \,\middle|\, b\varepsilon \bigcap_{n=1}^{2} \bigcap_{(a_1,a_2)\varepsilon R_n(a)} \Delta_n G(a_1, a_2)\right\}$$

Then $H=Q$.

Proof It is obvious from the definition of Q that $Q \subseteq H$. From the proposition, $H \subseteq Q$. Hence $H = Q$.

If the winnowing takes place successively, first constructing a relation T_2 from a given T_1 and then a T_3 from a given T_2 and so on, it is obvious that if $H \subseteq T_1$, then $H \subseteq T_n$ for each n, and for finite sized relations, there must be an N such that $T = T_n$ for all $n \geqq N$. In other words, the process converges in a finite number of steps to a fixed point.

Using the T in Figure 5 as T_1, Figure 6 shows how certain associations of the pairs of R with the pairs of S can be eliminated from consideration. As a result a new relation T_2 can be constructed which is contained in T_1 and T_2 contains all the homomorphisms T_1 contains. Figure 7 shows what happens if

	ac	ad	be	cb	cd	db	dc
13			/				
14			/				
32	/	/					
34	/	/	/				
43	/	/	/				
45	/	/					

T_2	Allowed Possibilities
1	acd
2	bcde
3	cd
4	cd
5	bcde

FIGURE 6 illustrates Proposition 2. Since $R \circ H \subseteq S$ implies $H(3) \subseteq \{b, c, d\}$, (see Figure 2), all possible relationships which allow 3 to be mapped to a or e are illegal. Hence (1, 3) cannot be mapped by H to (b, e); (3, 2) cannot be mapped to (a, c) or (a, d); (3, 4) cannot be mapped by H to (a, c) or (a, d); and (4, 3) cannot be mapped by H to (b, e). Since $H(4) \subseteq \{b, c, d\}$, (1, 4) and (3, 4) cannot be mapped by H to (b, e); (4, 3) and (4, 5) cannot be mapped by H to (a, c) or (a, d). All the illegal possibilities are marked out as per Proposition 2. Allowed possibilities are tabulated on the right.

	ac	ad	be	cb	cd	db	dc
13			/	/		/	
14			/	/		/	
32	/	/	/				
34	/	/	/	/		/	
43	/	/	/	/		/	
45	/	/	/				

T_3	Allowed Possibilities
1	acd
2	bcd
3	cd
4	cd
5	bcd

FIGURE 7 illustrates Proposition 2. Since $R \circ H \subseteq S$ implies $H(3)\{c, d\}$ (see Figure 3), (1, 3) or (4, 3) cannot be mapped to (c, b) or (d, b) and (3, 2) or (3, 4) cannot be mapped to (b, e). Since $H(4) \subseteq \{c, d\}$ (1, 4) or (3, 4) cannot be mapped to (c, b) or (d, b) and (4, 5) cannot be mapped to (b, e). The disallowed mappings are marked out. As per Proposition 2, allowed possibilities are tabulated on the right.

the winnowing process begins using the T_2 of Figure 6 to construct the smaller relation T_3 shown in Figure 7. Further iterations cannot reduce T_3 any more.

Unfortunately, the resulting relation T_3 is not single-valued and although it is a fixed point of the winnowing process, it does not necessarily have the composition property: $R \circ T_3 \subseteq S$. Hence, not all fixed points of the winnowing process are homomorphisms. Proposition 2 tells us which fixed points of the winnowing process are guaranteed to have the composition property: Any single-valued relation f which is a fixed point of the winnowing process is guaranteed to have the composition property $R \circ f \subseteq S$.

PROPOSITION 2 *Let* $R \subseteq A \times A$, $S \subseteq B \times B$. *Let* $f \subseteq A \times B$ *be single-valued.*

Define $G = \{(a_1, a_2, b_1, b_2) \varepsilon R \times S \mid (a_1, b_1) \varepsilon f, (a_2, b_2) \varepsilon f\}$

Suppose
$$f = \left\{(a, b) \varepsilon A \times B \mid b \varepsilon \bigcap_{n=1}^{2} \bigcap_{(a_1, a_2) \varepsilon R_n(a)} \Delta_n G(a_1, a_2)\right\}$$

Then $R \circ f \subseteq S$.

Proof Let $(b_1, b_2) \varepsilon R \circ f$. Then for some $(a_1, a_2) \varepsilon R$, $(a_1, b_1) \varepsilon f$ and $(a_2, b_2) \varepsilon f$. Now $(a_1, b_1) \varepsilon f$ implies

$$b_1 \varepsilon \bigcap_{n=1}^{2} \bigcap_{(\alpha_1, \alpha_2) \varepsilon R_n(a)} \Delta_n G(\alpha_1, \alpha_2).$$

Since, $(a_1, a_2) \varepsilon R_1(a_1)$, $b_1 \varepsilon \Delta_1 G(a_1, a_2)$.

Obviously $G(a_1, a_2) \neq \emptyset$ so let $(\beta_1, \beta_2) \varepsilon G(a_1, a_2)$. Now $(a_1, a_2, \beta_1, \beta_2) \varepsilon G$ implies $(\beta_1, \beta_2) \varepsilon S$ and $(a_1, \beta_1) \varepsilon f$ and $(a_2, \beta_2) \varepsilon f$. But f is single-valued so that $(a_1, \beta_1) \varepsilon f$ and $(a_1, b_1) \varepsilon f$ imply $b_1 = \beta_1$. Likewise $(a_2, \beta_2) \varepsilon f$ and $(a_2, b_2) \varepsilon f$ imply $b_2 = \beta_2$. Then, $(b_1, b_2) = (\beta_1, \beta_2) \varepsilon S$ so that $R \circ f \subseteq S$.

Thus our search for homomorphisms is now limited to trying to determine all the single-valued fixed points of the winnowing process. It would be nice if we could take two or more fixed points of the winnowing process, somehow combine them together in an appropriate way, and then have the resulting relation also be a fixed point of the process. With such a mechanism, it might be possible to generate the single-valued fixed points of the winnowing process from some small set of easily determined fixed points of the winnowing process.

There is a natural place to look for the easily determined fixed points of the winnowing process. Suppose we begin the process with a relation which allows everything in A to be paired with everything in B except that the element $a_1 \varepsilon A$ is allowed to be paired only with the element $b_1 \varepsilon B$. The successive winnowing process then will determine a fixed point, called the basis relation $T^{a_1 b_1}$, which contains

	ac	ad	be	cb	cd	db	dc
13			/	/	/	/	/
14			/	/	/	/	/
32							
34							
43							
45							

T_1^{1a}	
1	a
2	abcde
3	abcde
4	abcde
5	abcde

	ac	ad	be	cb	cd	db	dc
13			/	/	/	/	/
14			/	/	/	/	/
32	/	/	/				
34	/	/	/	/		/	
43	/	/	/	/		/	
45	/	/	/				

T_2^{1a}	
1	a
2	bcd
3	cd
4	cd
5	bcd

	ac	ad	be	cb	cd	db	dc
13			/	/	/	/	/
14			/	/	/	/	/
32	/	/	/				
34	/	/	/	/		/	
43	/	/	/	/		/	
45	/	/	/				

T_3^{1a}	
1	a
2	bcd
3	cd
4	cd
5	bcd

FIGURE 8 shows the successive eliminations of mapping possibilities by the construction of T^{1a} by successive winnowing.

all homomorphisms which map a_1 to b_1. Figure 8 illustrates the results of successive winnowing where the element 1 is constrained to map to a.

The intersection of $T^{a_1b_1}$ with another basis relation $T^{a_2b_2}$ will contain all homomorphisms mapping a_1 to b_1 and a_2 to b_2.

Intersections of basis relations where the intersections stay defined everywhere then become candidates for homomorphisms. Proposition 3 states that if f is any defined everywhere relation from A to B and f has the representation

$$\bigcap_{(a,b)\varepsilon f} T^{ab},$$

where T^{ab} is any fixed point of the winnowing process which maps a to only b, then f is a fixed point of the winnowing process.

PROPOSITION 3 *Let* $R\subseteq A\times A$, $S\subseteq B\times B$, $f\subseteq A\times B$.

For each $(a, b)\varepsilon A\times B$ let $T^{ab}\subseteq A\times B$ satisfy

1) $(a,\beta)\varepsilon T^{ab}$ implies $b=\beta$

2) $$T^{ab}=\left\{(\alpha,\beta)\varepsilon A\times B \,\middle|\, \beta\varepsilon \bigcap_{n=1}^{2} \bigcap_{(a_1,a_2)\varepsilon R_n(\alpha)} \Delta_n G^{ab}(a_1,a_2)\right\}$$

where

$$G^{ab}=\{(a_1,a_2,b_1,b_2)\varepsilon R\times S \mid (a_1,b_1)\varepsilon T^{ab} \text{ and } (a_2,b_2)\varepsilon T^{ab}\}$$

If f is defined everywhere and

$$f=\bigcap_{(a,b)\varepsilon f} T^{ab},$$

then f is a fixed point of the winnowing process; that is if

$$G=\{(a_1,a_2,b_1,b_2)\varepsilon R\times S \mid (a_1,b_1)\varepsilon f \text{ and } (a_2,b_2)\varepsilon f\}$$

and $$f'=\left\{(\alpha,\beta)\varepsilon f \,\middle|\, \beta=\bigcap_{n=1}^{2} \bigcap_{(a_1,a_2)\varepsilon R_n(\alpha)} \Delta_n G(a_1,a_2)\right\}$$

then $f'=f$.

Proof By Proposition 1, $f'\subseteq f$ so all we need show is that $f\subseteq f'$. Let $(\alpha,\beta)\varepsilon f$. We want to show that

$$\beta \bigcap_{n=1}^{2} \bigcap_{(a_1,a_2)\varepsilon R_n(\alpha)} \Delta_n G(a_1,a_2)$$

Either $R_n(\alpha)=\emptyset$, $n=1, 2$ or not. If $R_n(\alpha)=\emptyset$, $n=1, 2$ then

$$\bigcap_{n=1}^{2} \bigcap_{(a_1,a_2)\varepsilon R_n(\alpha)} \Delta_n G(a_1,a_2)=B$$

so that the assertion is trivially true.

If $R_n(\alpha)\neq\emptyset$ for some n, then $R_1(\alpha)\neq\emptyset$ or $R_2(\alpha)\neq\emptyset$. Suppose $R_1(\alpha)\neq\emptyset$. Then let $(a_1, a_2)\varepsilon R_1(\alpha)$. Since f is defined everywhere, there exists a γ such that $(a_2,\gamma)\varepsilon f$. But

$$f=\bigcap_{(a,b)\varepsilon f} T^{ab}$$

so that surely $(\alpha,\beta)\varepsilon T^{a_2\gamma}$. Hence, by definition of $T^{a_2\gamma}$,

$$\beta\varepsilon \bigcap_{n=1}^{2} \bigcap_{(\alpha_1,\alpha_2)\varepsilon R_n(\alpha)} \Delta_n G^{a_2\gamma}(\alpha_1,\alpha_2)$$

But $(a_1, a_2)\varepsilon R_1(\alpha)$ so that we must have $\beta\varepsilon\Delta_1 G^{a_2\gamma}(a_1,a_2)$. Hence, there exists a δ such that $(\beta, \delta)\varepsilon G^{a_2\gamma}(a_1,a_2)$. Thus $(\beta,\delta)\varepsilon S$, $(a_1,\beta)\varepsilon T^{a_2\gamma}$, and $(a_2,$

$\delta)\ T^{\alpha_2\gamma}$. But $(a_2, \delta)\varepsilon T^{a_2\gamma}$ implies $\delta = \gamma$ so we have $(\beta, \gamma) = (\beta, \delta)\varepsilon S$.

Now, $(\beta, \gamma)\varepsilon S$, $(a_1, \beta)\varepsilon f$, $(a_2, \gamma)\varepsilon f$, and $(a_1, a_2)\varepsilon R$ imply $(\beta, \gamma)\varepsilon G(a_1, a_2)$ so that

$$\beta \varepsilon \Delta_1 G(a_1, a_2).$$

Since this is true for each $(a_1, a_2)\varepsilon R_1(\alpha)$,

$$\beta \varepsilon \bigcap_{(a_1, a_2)\varepsilon R_n(\alpha)} \Delta_1 G(a_1, a_2).$$

If $R_2(\alpha) \neq \emptyset$, a similar argument shows

$$\beta \varepsilon \bigcap_{(a_1, a_2)\varepsilon R_2(\alpha)} \Delta_2 G(a_1, a_2)$$

Hence

$$\beta \varepsilon \bigcap_{n=1}^{2} \bigcap_{(a_1, a_2)\varepsilon R_n(\alpha)} \Delta_n G(a_1, a_2).$$

3 BINARY RELATION HOMOMORPHISM CHARACTERIZATION

From Proposition 3 it is only a short way to a characterization of the binary relation homomorphism for if T^{ab} is in fact not just any fixed point of the winnowing process which allows a to be mapped only to b, but if T^{ab} is the basis relation determined by the winnowing process which begins with $\{(a, b)\} \cup (A - \{a\}) \times B$, then we should certainly have

$$H \subseteq \bigcap_{(a, b)\varepsilon H} T^{ab}$$

for any homomorphism H, from Proposition 1; and since a homomorphism is defined everywhere, the form

$$\bigcap_{(a,b)\varepsilon H} T^{ab}$$

must be single-valued so that

$$H \supseteq \bigcap_{(a,b)\varepsilon H} T^{ab}.$$

This gives us

$$H = \bigcap_{(a,b)\varepsilon H} T^{ab}$$

for any homomorphism H.

Conversely if H is any defined everywhere binary relation having the representation

$$H = \bigcap_{(a,b)\varepsilon H} T^{ab},$$

Proposition 3 states that H is a fixed point of the winnowing process since H is defined everywhere. And since $(a, b')\varepsilon T^{ab}$ implies $b' = b$, H defined everywhere implies

$$\bigcap_{(a,b)\varepsilon H} T^{ab}$$

is single-valued. By Proposition 2,

$$\bigcap_{(a,b)\varepsilon H} T^{ab}$$

single-valued and a fixed point of the winnowing process imply that

$$\bigcap_{(a,b)\varepsilon H} T^{ab}$$

has the composition property

$$R \circ \bigcap_{(a,b)\varepsilon H} T^{ab} \subseteq S.$$

And since

$$\bigcap_{(a,b)\varepsilon H} T^{ab}$$

is defined everywhere, single-valued, and has the composition property, it must be a homomorphism.

THEOREM (Binary Relation Homomorphism Characterization Theorem). *Let $R \subseteq A \times A$, $S \subseteq B \times B$, and $H \subseteq A \times B$. For each (a, b) $A \times B$ let*

$$T_1^{ab} = \{(a, b)\} \cup (A - \{a\}) \times B.$$

Define $T_2^{ab}, \ldots, T_n^{ab}, \ldots$ iteratively by

$$T_{n+1}^{ab} = \left\{ (\alpha, \beta)\varepsilon T_n^{ab} \,\middle|\, \beta = \bigcap_{n=1}^{2} \bigcap_{(a_1, a_2)\varepsilon R_n(\alpha)} \Delta_n G^{ab}(a_1, a_2) \right\}$$

where

$$G^{ab} = \{(a_1, a_2, b_1, b_2)\varepsilon R \times S \mid (a_1, b_1)\varepsilon T_n^{ab} \text{ and } (a_2, b_2)\varepsilon T_n^{ab}\}.$$

Suppose for all $n \geqq N$ and for all $(a, b)\varepsilon A \times B$, $T^{ab} = T_n^{ab}$. H is a homomorphism of R into S if and only if

1) $\bigcap_{(a,b)\varepsilon H} T^{ab}$ *is defined everywhere*

2) $H = \bigcap_{(a,b)\varepsilon H} T^{ab}$

Proof Suppose H is a homomorphism of R into S. Let $(\alpha, \beta)\varepsilon H$. Then by Proposition 1, $H \subseteq T^{\alpha\beta}$. Since this is true for each

$$(\alpha, \beta)\varepsilon H, H \subseteq \bigcap_{(\alpha,\beta)\varepsilon H} T^{\alpha\beta}.$$

Now let

$$(a, b)\varepsilon \bigcap_{(\alpha,\beta)\varepsilon H} T^{\alpha\beta}.$$

Since H is defined everywhere, there exists a γ such that $(a,\gamma)\varepsilon H$. Then

$$(a, b)\varepsilon \bigcap_{(\alpha,\beta)\varepsilon H} T^{\alpha\beta}$$

implies $(a,b)\varepsilon T^{a\gamma}$. But by construction of $T^{a\gamma}$. $(a, b)\varepsilon T^{a\gamma}$ implies $b=\gamma$. Hence, $(a,b)=(a,\gamma)\varepsilon H$ so that

$$\bigcap_{(\alpha,\beta)\varepsilon H} T^{\alpha\beta} \subseteq H.$$

Now

$$H \subseteq \bigcap_{(\alpha,\beta)\varepsilon H} T^{\alpha\beta}$$

and

$$\bigcap_{(\alpha,\beta)\varepsilon H} T^{\alpha\beta} \subseteq H$$

imply

$$\bigcap_{(\alpha,\beta)\varepsilon H} T^{\alpha\beta} = H.$$

And since H is a homomorphism, it is defined everywhere so that

$$\bigcap_{(\alpha,\beta)\varepsilon H} T^{\alpha\beta}$$

is defined everywhere.

Suppose

$$H = \bigcap_{(a,b)\varepsilon H} T^{\alpha\beta}$$

is defined everywhere. By Proposition 3, H is a fixed point of the winnowing process. Also H is single-valued for if $(a, b_1)\varepsilon H$ and $(a, b_2)\varepsilon H$ then $(a, b_1)\varepsilon T^{ab_2}$ which implies $b_1 = b_2$. By Proposition 2, any single-valued relation H which is a fixed point of the winnowing process has the composition property $R \circ H \subseteq S$. Now by definition of homomorphism, H is a homomorphism of R into S.

4 DEPTH FIRST SEARCH

The binary relation characterization theorem allows all homomorphisms of R into S to be found by a depth first search in the following manner. Suppose we are looking for homomorphisms which map the element $1\varepsilon A$ to the element $a\varepsilon B$. We can determine by the successive winnowing process the basis relation T^{1a} which must contain all such homomorphisms. Now, T^{1a} may have other elements of A which are uniquely mapped to elements of B. If so, we can determine the basis relation for these pairs and take the intersection of all of them with T^{1a}. The resulting intersection must contain any homomorphism which maps 1 to a. If the resulting intersection relation has additional elements which are uniquely mapped, more intersections can be taken. When the resulting intersection has no more additional elements which are uniquely mapped, then one of three cases exists:

1) either the intersection is not defined everywhere in which case no homomorphism mapping 1 to a exists;

2) or the intersection relation f is defined everywhere and single-valued in which case

$$f = \bigcap_{(a,b)\varepsilon f} T^{ab}$$

so that f is a homomorphism;

3) or the intersection relation is defined everywhere and not single-valued in which case a choice must be made in a branch of the depth first search.

The choice is to map to a unique element of B one of those elements of A having possible multiple associations with the elements of B. In this last case, once such a choice is made, the corresponding basis relation must be intersected with the previously intersected relations. This brings us back to the point of looking for additional uniquely mapped pairs. From here the search iterates until each branch of the tree terminates in one of the first two cases.

Figure 9 lists all the basis relations which are defined everywhere for our example relation R and S. Figure 10 illustrates the tree determined by a depth first search. The tree shows all the 8 possible relation homomorphisms which map 1 to a. Figure 11 shows the 8 homomorphisms and their corresponding homomorphic images.

A simple specialization of this iterative process allows the determination of whether one relation is isomorphic to a part of another. In the depth first search, terminate any branch when two elements

	T^{1a}	T^{2b}	T^{2c}	T^{2d}	T^{3c}	T^{3d}	T^{4c}	T^{4d}	T^{5b}	T^{5c}	T^{5d}
1	a	acd	a	a	a	a	a	a	acd	a	a
2	bcd	b	c	d	bd	bc	bc	bd	bcd	bd	bc
3	cd	cd	d	c	c	d	d	c	cd	c	d
4	cd	cd	c	d	d	c	c	d	cd	d	c
5	bcd	bcd	bd	bc	bc	bd	bd	bc	b	c	d

FIGURE 9 tabulates those basis relations which are defined everywhere.

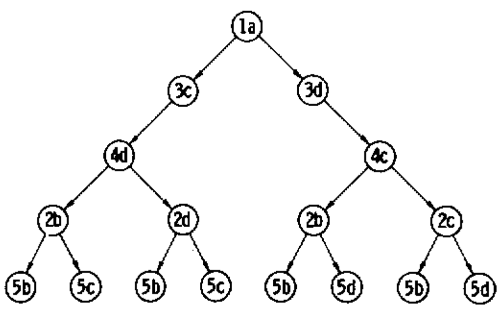

FIGURE 10 illustrates a tree determined by a depth first search. The tree shows all the 8 possible relation homomorphisms which map 1 to A.

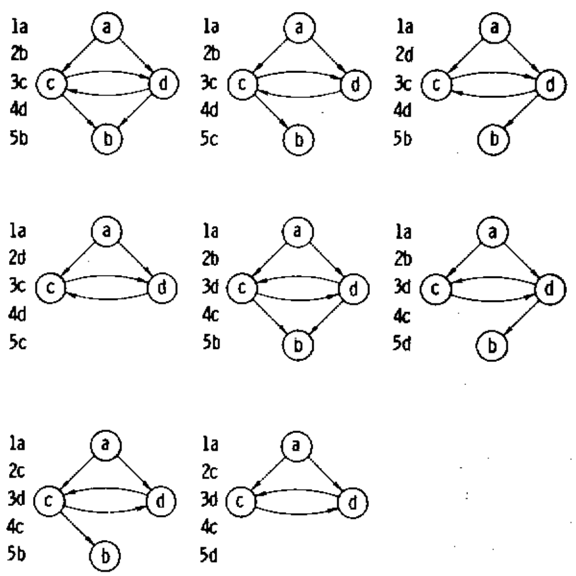

FIGURE 11 illustrates the 8 possible homomorphisms and their corresponding homomorphic images.

from A map to the same element from B. This guarantees that the resulting intersection relation will be one–one. Since we assumed the relations R and S to be finite, one–one homomorphisms are isomorphisms.

5 NUMBER OF OPERATIONS

A crude but simple estimate can be made of the average number of operations needed to calculate all the homomorphisms from relation R to relation S assuming that the number of intersections which need to be taken in the depth first search is proportional to the number of homomorphisms which exist. This assumption is not valid for pathological worst cases, but it is probably alright for more typical cases. Suppose that $R \subseteq A \times A$ and $S \subseteq B \times B$. Let $\#$ be the counting measure. Suppose

$$\#R = L_1 \qquad \#S = L_2$$
$$\#A = N_1 \qquad \#B = N_2$$

$\#\{H \subseteq A \times B | H$ is defined everywhere, single-valued, and $R \circ H \subseteq S\} = K$.

Then, there are $N_1 N_2$ basis relations to determine. For each basis relation the winnowing process itself can take no more than $N_1 N_2$ iterations to converge to a fixed point. Each winnowing iteration cannot take more than $L_1 L_2 (L_1 + L_2) N_1$ operations. Thus the basis relations require at most $N_1^3 N_2^2 L_1 L_2 (L_1 + L_2)$ operations to calculate them. Each basis relation has at most $N_1 N_2$ members and the intersection of the two basis relations stored in an ordered manner requires at most $2N_1 N_2$ operations. In the forest containing trees like that of Figure 10, there are at most $N_1 N_2$ trees. Each tree has complete branches which are N_1 levels long. There must be a total of no more than K complete tree branches in the entire forest. There are an average of $N_1 K$ nodes at which intersections must be taken. Since each relation intersection takes no more than $2N_1 N_2$, operations we obtain that the total number of operations required to find all homomorphisms is fixed cost of no more than $2N_1^3 N_2^2 L_1 L_2 (L_1 + L_2)$ operations and an average number $N_1^2 N_2 K$ of operations for the depth first search.

6 CONCLUSION

We have shown how all relation homomorphisms have a representation in the form of the intersection of some basis relations and how it is that all intersections of basis relations taken over any relation defined everywhere homomorphisms when the resulting intersection relation is defined everywhere. Using this fact, we have suggested a depth first search procedure for generating all the homomorphisms of one relation to another. A rough estimate on the number of operations involved in this process is proportional to the number of homomorphisms that exists from one relation to the other times the number of nodes cubed plus a fixed overhead cost proportional to the number of pairs in the relation cubed times the number of nodes to the fifth power.

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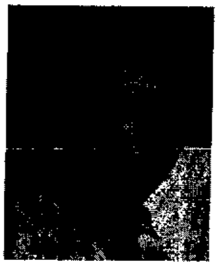
REFERENCES

1. A. T. Berztiss, "A Backtrack Procedure for Isomorphism of Directed Graphs." *J. ACM*, **20**, No. 3, July 1973, pp. 365–377.
2. D. G. Corneil and C. C. Gotlieb, "An Efficient Algorithm for Graph Isomorphism." *J. ACM* **17**, No. 1, January 1970, pp. 51–64.
3. F. Harary, *Graph Theory*. Addison–Wesley, Reading, Massachusetts, 1969.
4. J. R. Ullman, "An Algorithm for Subgraph Isomorphism." *J. ACM*, **23**, No. 1, January, 1976, pp. 31–42.
5. S. H. Unger, "GIT—A Heuristic Program for Testing Pairs of Directed Line Graphs for Isomorphism." *Comm. ACM*, **7**, No. 1, January 1964, pp. 26–34.
6. D. L. Waltz, "Automata Theoretic Approach to Visual Processing." *Applied Computation Theory*, edited by R. T. Yeh, Prentice–Hall, Englewood Cliffs, New Jersey, 1975.
7. D. L. Waltz, "Generating Simantic Descriptions from Drawings of Scenes with Shadows." *MIT Technical Report A1271*, November, 1972.



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