Multi-resolution Morphology

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Abstract

The speed and storage advantages of performing vision tasks at downsampled levels of multiresolution data structures can incur the cost of distortion caused by the sampling process. We analyze the constraints on sampling and on image objects, in order to speed up morphological operations without sacrificing accurate shape analysis. The following results are shown to be true under reasonable morphological sampling conditions. Sets which are sampled can be reconstructed in two ways: either a closing or dilation. In both reconstructions, the sampled reconstructed sets are equal to the sampled sets. A set contains its reconstruction by closing and is contained in its reconstruction by dilation; indeed, these are extremal bounding sets. That is, the largest set which downsamples to a given set is its reconstruction by dilation; the smallest is its reconstruction by closing. Furthermore, the distance from the maximal reconstruction to the minimal reconstruction is no more than the diameter of the reconstruction structuring element. Morphological sampling thus provides reconstructions positioned only to within some spatial tolerance, in contrast to sampling in signal processing which reconstructs only those frequencies bandlimited to Nyquist frequency. All sampling results in the binary case generalize to the grayscale case. For greyscale, the boundedness constraint on the reconstruction translates to a boundedness both spatially and in the greyscale image. This work provides a basis for multi-resolution shape analysis in shape-based hierarchical structures, supplementing the advantages of multi-resolution techniques with those of morphological operators on cellular arrays.

I. Introduction

Multi-resolution techniques [Ahuja & Swamp, Klüver, Mersereau & Speake, Tanimoto, Uhr] have been useful for at least two fundamental reasons: the representation they provide naturally focuses a computational mechanism on objects or features likely to be at least some specified size [Crowley, Miller & Stout, Rosenfeld, Witkin]; and a significant reduction in the required number of operations comes from using only those levels in the resolution hierarchy whose resolution suffices for the detection and localization of specified-size objects or features, instead of full resolution [Burt, Dyer, Lougheed & McCubrey]. The resolution hierarchy, called a pyramid, is produced typically by low pass filtering and then sampling to generate the next lower resolution level. The purpose of the low pass filter is to remove from the higher resolution image those spatial frequencies which are higher than the Nyquist frequency corresponding to the sampling spacing. It is well known in signal processing that if sampling is not preceded by this kind of filtering, the higher frequency energy manifests itself in extraneous lower frequency patterns in the sampled data, an undesired phenomenon called aliasing. Figure 1 shows a 5-level pyramid produced by pure sampling from a laser range image. The highest resolution image size is 256 X 256. A 2-pixel wide line and a 4 X 4 box are placed intentionally at the upper right and upper left, respectively. Figure 2 shows the 5-level pyramid produced by a 3 X 3 box filtering followed by sampling to generate the next pyramid level. Because multi-resolution pyramids are used for detection and identification of objects or features of at least a specified size, it is natural to ask if mathematical morphology [Matheron, Serra, Sternberg, Haralick et al] might provide a better basis than low pass filters, for constructing pyramids. This question is suggested by the fact that mathematical morphology deals directly with shape, whereas low pass filtering techniques are based on linear combinations of sinusoidal waveforms, a representation far removed from shape. A basis for morphological pyramids requires a morphological sampling theorem to analyze the relationship between the unsampled image and an appropriately (morphologically) filtered and sampled image. Such a theorem must explain which kinds of shapes are preserved and which are suppressed or eliminated, from one pyramid level to the next. Performance of a less costly morphological filtering operation on the sampled image must be related to performance of the more costly equivalent on a higher resolution image. It is just these issues which we address in this paper. In section II we review the basic definitions and properties for binary morphology operations. In section III, we initially develop the morphological sampling theorem for binary morphology. The homomorphism theorem between binary and greyscale morphology implies that each result in binary morphology has a corresponding result in greyscale morphology. Section IV contains these greyscale generalizations.

II. Preliminaries

Let $E$ denote the set of numbers used to index a row or column position on a binary image. The binary image itself can then be thought of as a subset of $E$. Pixels are in this subset if and only if they have the binary value one on the image. This correspondence permits us to work with sets rather than with image functions, indeed, with sets in $E^n$. The first two operations of mathematical morphology are the dual operations of dilation and erosion. The dilation of a set $A \subseteq E^n$ with a set $B \subseteq E^n$ is defined by

$$A \oplus B = \{x | \forall y \in E^n (B \subseteq x \rightarrow (x + y) \subseteq A)\}$$

For any set $A \subseteq E^n$ and $x \in E^n$, let $A_x$ denote the translation of $A$ by $x$;

$$A_x = \{y | \forall a \in A, y + a = x\}$$

For any set $A \subseteq E^n$, let $A$ denote the reflection of $A$ about the origin;

$$A = \{x | \forall a \in A, x = -a\}$$

Relationships satisfied by dilation and erosion include the following:

$$A \oplus B = B \oplus A$$

$$A \ominus B = A \oplus (-B)$$

$$A \oplus B = A \oplus (B \ominus A)$$

$$A \ominus B = A \ominus (B \oplus A)$$

$$A \ominus B = A \ominus (A \ominus (B \ominus A))$$

$$A \ominus B = \bigcup_{x \in B} A_x$$

$$A \ominus B = \bigcap_{x \in B} A_x$$
The opening of A by B is defined by
\[ A \ominus B = (A \ominus B) \ominus B. \]

The closing of A by B is defined by
\[ A \oplus B = (A \oplus B) \oplus B. \]

Opening and closing satisfy the following basic relationships.

\[ (A \ominus B) \circ B = A \ominus B \]
\[ (A \ominus B) \bullet B = A \ominus B \]
\[ A \ominus B \subseteq A \ominus (A \ominus B) \]
\[ A \ominus B = A \ominus C \subseteq B \ominus C \]
\[ (A \ominus B)^* = A^* \ominus B \]

The reason that openings and closings deal directly with shape properties is apparent from the following representation theorem for openings.

\[ A \ominus B = \{x | \text{for some } y, x \in B \ominus y \subseteq A \}. \]

A opened by B contains only those points of A which can be covered by some translation of B, which is, in turn, entirely contained inside A. Thus x is a member of the opening if it lies in some area inside A which entirely contains a translated copy of the shape B. In this sense, A opened by B is the set of all points of A which can participate in areas of A which match B. If B is a disk of diameter d, for example, then A \ominus B would be that part of A which in no place is narrower than d. Figure 3 shows a greyscale opening applied to a noisy checkerboard image (a) is the original image, (b) is the noisy image, and (c) is the opened result.

The duality relationship \((A \ominus B)^* = A^* \ominus B\) between opening and closing implies a corresponding representation theorem for closing.

\[ A \oplus B = \{x | \text{for some } y, x \in B \oplus y \subseteq A \}. \]

A closed by B consists of all those points x for which x being covered by some translation of B implies that B hits or intersects some part of A.

One natural way to reconstruct a sampled opening is by dilation. If S and K are coordinated to make the reconstructed image (dilated, sampled, opened image) the same as the opened image, we would have a morphological sampling theorem nearly identical to the standard sampling theorem of signal processing. However, morphology cannot provide a perfect reconstruction, as is illustrated by the following one-dimensional example. Let the image F be the union of three open intervals \(F = (3.1, 7.4) \cup (11.5, 11.6) \cup (18.9, 19.8)\), where (x, y) denotes the open interval between x and y. We can remove all details of less than length 2 by opening with the structuring element \(K = (-1, 1)\). Then the opened image \(F \ominus K = (3, 7.4)\). What should the corresponding sample set be? Consider a sample set \(S = \{x | x \text{ is an integer}\}\), with a sample spacing of unity; spacings of .2, .5 or .7 would illustrate the same difficulty as well. The sampled opened image \((F \ominus K) \cap S = 4, 5, 6, 7\) Dilating by K to reconstruct the image produces \((F \ominus K) \cap S = 4, 5, 6, 7\), an incorrect reconstruction. The dilation fills in between the sample points, but cannot "know" to proceed left by a length of .9 and yet by 4 to the right. However in this case the reconstruction is the largest one for which sampling the reconstruction produces the sampled opened image \((F \ominus K) \cap \cap S\neq (3, 4, 5, 6, 7, 8)\) does not bandlimit the image. In fact, when the image is piecewise continuous the sampled opened image belongs to a special class of infinite bandwidth signals, wherein reconstructing the sampled opened image as specified by the sampling theorem cannot produce the kind of aliasing found in Moiré patterns. The standard sampling theorem produces a bandlimited reconstruction which passes through the sample points. Thus, step-like patterns like the open intervals of F get reconstructed with ringing throughout and with overshoot and undershoot at step edges. By contrast, the morphological reconstruction cannot produce ringing, but the position of any step edge is uncertain within the sampling interval. Figure 5 shows a 5-level pyramid produced, from the same image as in Figure 1, by first opening with a 2 X 3 brick, then sampling an F reconstructed, before finally downsampling to generate the next level. In the remainder of this section we give a complete derivation of the results illustrated in the example. First note that to use a structuring element K as a "reconstruction kernel", K must be large enough to ensure that the dilation of the sampling set S by K covers the entire space \(\mathbb{R}^n\). For technical reasons apparent in the derivations, we also require that K be symmetric, \(K = \tilde{K}\). Under these general conditions the erosion and dilation of the original image F by K bound the reconstructed sampled image.

Proposition: Let \(F, K, S \subseteq \mathbb{R}^n\). Suppose \(S \cap K = \emptyset\) and \(K = \tilde{K}\). Then
\[ F \ominus K \subseteq (F \cap S) \ominus K \subseteq F \ominus K. \]

Proof: Since \(F \cap S \subseteq K\), \(F \cap S \cap K \subseteq F \cap K\). To show \(F \ominus K \subseteq (F \cap S) \ominus K\), we show \((F \cap S) \ominus K \subseteq (F \ominus K)\). If \(z \in (F \cap S) \ominus K\), then \(F \cap S \supseteq K\). Hence, if \(k \in K\), \(x+k \in F\). Therefore, \(z \neq x+k \notin K\). Thus there exists \(k \in K\), \(z+x+k \in F\), and \(z \circ k \notin K\). Therefore, \(z \circ k \notin K\). Thus there exists \(k \in K\), \(z \circ k \notin K\).

In the standard sampling theorem, the period of the highest frequency must be sampled at least twice in order to reconstruct the signal from the sampled form. In mathematical morphology there is analogous requirement. The smallest detail an opening by K permits is the size of K. We express the requirement that K be sampled at least twice, by the slightly weaker "sampling condition" that \(z \notin K \Rightarrow K \cap \gamma(z) \neq \emptyset\). If K is not empty, this condition that \(S \cap K = \emptyset\). If the points in the sampling set S are spaced no further than 2d apart, then the corresponding kernel K can be the open ball of radius 2d. In this case \(z \notin K \Rightarrow K \cap \gamma(z) \neq \emptyset\), but this intersection may contain only one sampling point. Note that we do not require that S be regularly sampled.
spaced, but only that for every \( x \in \mathbb{E}' \), there exists some \( s \in S \) such that \( \|x - s\| < \varepsilon \). If this is the case, then whenever \( |a - b| < 2\varepsilon \) there is always \( s \in S \) such that \( |a - s| < 2\varepsilon \) and \( |x - s| < 2\varepsilon \).

Now we can show that if a symmetric kernel satisfies the sampling condition, then any set which is open under the kernel is contained in its reconstructed sampling.

**Proposition:** Let \( F, K, S \subseteq \mathbb{E}' \), \( x \in K \Rightarrow K_s \cap K_t \cap S \neq \phi \), \( K = K_s \), and \( F = F_0 \). Then \( F \subseteq (F \cap S) \circ K \).

Proof: Let \( x \in F \). Since \( F = F_0 \circ K \), there exists \( y \) such that \( x \in K \subseteq F \). Then \( K_s \cap K_t \cap S \neq \phi \), and there exists \( s \) such that \( s \in K \cap K_t \cap S \).

If \( s \in S \), then \( s \in (F \cap S) \circ K \).

Consider \( F = (3.1, 7.4) \cup (11.5, 11.6) \cup (18.9, 19.8) \),

Then \((F \cap S) \circ K \) is maximal with respect to the property of downsampling.

The condition that \( F \) is open under \( K \) is not too large with respect to \( S \), and \( F \) satisfies the condition that \( F \) is not open under \( K \).

Now we can show that \( F \) is open under \( K \).

**Proposition:** Let \( F, K, S \subseteq \mathbb{E}' \). If \( F \cap K = K \), then \((F \cap S) \circ K \) is open under \( K \).

Proof: If \( F \cap K = F \), then \((F \cap S) \circ K \subseteq (F \cap S) \circ K \).

Since it is always true that \( A \subseteq A \), this is the case, then whenever \( l \leq 2 \).

These results constitute the binary morphological sampling theorem: if \( S, F, K \subseteq \mathbb{E}' \) are respectively a sampling set, reconstruction structure element, and open set satisfying

\[
\begin{align*}
(1) &\ K = K_s, \\
(2) &\ K \cap S = (0), \\
(3) &\ \exists \ s \in K \cap K_t \cap S \neq S, \\
(4) &\ (F \cap S) \circ K \subseteq (F \cap S) \circ K, \\
(5) &\ (F \cap S) \circ K \subseteq (F \cap S) \circ K, \\
(6) &\ F \cap K = F.
\end{align*}
\]

Theorem: Let \( F \subseteq \mathbb{E}' \) be a sampling set, and \( K \subseteq \mathbb{E}' \) be the reconstruction structure element for the sampling, satisfying

\[
\begin{align*}
(1) &\ K \cap S = (0), \\
(2) &\ S \cap S \neq S.
\end{align*}
\]

Then \((F \cap S) \circ K \cap K \subseteq F \cap S \).

Proof: \((F \cap S) \circ K \cap K = (F \cap S) \circ K \cap K \).

The condition (1) is satisfied by our example \( F = (3.1, 7.4) \cup (11.5, 11.6) \cup (18.9, 19.8) \).

Then \((F \cap S) \circ K \cap K \subseteq F \cap S \).

\(\square\)

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Therefore the standard sampling theorem cannot produce a reconstruction with frequencies higher than the Nyquist frequency, the morphological sampling theorem cannot produce a reconstruction whose positional accuracy is better than the size of \( K \).

Thus we can see immediately by noticing that \((F \cap S) \circ K \subseteq (F \cap S) \circ K \).

The lower bound \((F \cap S) \circ K \) is a maximal reconstruction set \((F \cap S) \circ K \).

The upper bound \((F \cap S) \circ K \) is a maximal reconstruction set \((F \cap S) \circ K \).

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IV. Greyscale sampling theorem

In this section we present the extension of the morphological sampling theorem from the binary case to the greyscale case. We begin by reviewing some definitions and results [Haralick et al.].

We adopt the convention that the first \((N-1)\) coordinates of the \(N\)-tuples in a set \( A \subseteq \mathbb{E}' \) constitute the spatial domain of \( A \), and the \( N\)th coordinate represents the surface, i.e. \( A \subseteq \mathbb{E}'-1 \times \mathbb{E} \).

For greyscale images, \( N = 3 \) and an image is a function \( f : \mathbb{E} \times E \rightarrow \mathbb{E} \). A set \( A \subseteq \mathbb{E}'-1 \times \mathbb{E} \) is an umbra if and only if \( (x, y) \in A \) implies that \( (x, y) \in B \) for every \( y \in E \).

The top of \( A \) is a function \( T[A] \) mapping the spatial domain of \( A \), \( (x, y) \in \mathbb{E}'-1 \times \mathbb{E} \), to the "top surface" of \( A \).
the umbrella homomorphism theorem states that the operation of taking an umbrella is a homomorphism from the grey-scale morphology to the binary morphology. That is, if \( F, K \subseteq \mathbb{E}^{n-1} \), \( f : F \to E \), and \( k : K \to E \), the dilation of \( f \) by \( k \), is the mapping \( f \circ k : F \circ K \to E \) defined by

\[
  f \circ k = T[(f \cup k)]
\]

Similarly, the erosion of \( f \) by \( k \), maps \( F \circ K \to E \) by

\[
  f \circ k = T[(f \cap k)]
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The umbrella homomorphism theorem states that the operation of taking an umbrella is a homomorphism from the grey-scale morphology to the binary morphology. That is, if \( F, K \subseteq \mathbb{E}^{n-1} \), \( f : F \to E \), and \( k : K \to E \), the dilation of \( f \) by \( k \), is the mapping \( f \circ k : F \circ K \to E \) defined by

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  f \circ k = T[(f \cap k)]
\]

Furthermore, the operations of max and union, and of min and intersection, are homomorphic under the umbrella transformation. That is,

\[
  (1) U[f \circ k] = U[f] \circ U[k] \text{ and } \\
  (2) U[f \circ k] = U[f] \cap U[k]
\]

To be able to state the corresponding grey-scale results we need two more notational conventions. If \( S \subseteq \mathbb{E}^{n-1} \), and \( f : F \to E \), then the sampled version of \( f \) is the restriction of \( f \) to \( S \), denoted \( f|_S \). Hence \( f|_S : F \cap S \to E \) defined by \( f|_S(x) = f(x) \) for \( x \in F \cap S \). If \( k : K \to E \), then \( k \) is the reflection of \( k \), defined by \( k : K \to E \) where \( k(x) = k(-x) \).

The corresponding grey-scale morphology results are

\[
  (1) \text{Let } F, K, S \subseteq \mathbb{E}^{n-1}, f : F \to E, \text{ and } k : K \to E. \text{ Suppose } S \circ K = \mathbb{E}^{n-1} \text{ and } k = k. \text{ Then } \\
  f \circ k \leq f|_S \circ k \leq f \circ k
\]

\[
  (2) \text{Let } F, K, S \subseteq \mathbb{E}^{n-1}, f : F \to E, \text{ and } k : K \to E. \text{ Suppose } S \circ K = \mathbb{E}^{n-1}, k \in K, \text{ and } S \cap K = \emptyset. \text{ Then } \\
  f \circ k \leq f|_S \circ k
\]

\[
  (3) \text{Let } F, K, S \subseteq \mathbb{E}^{n-1}, f : F \to E, \text{ and } k : K \to E. \text{ Suppose } S \circ K = \mathbb{E}^{n-1}, k \in K, \text{ and } S \cap K = \emptyset. \text{ Then } \\
  f \circ k \leq f|_S \circ k
\]

\[
  (4) \text{Let } F, K, S \subseteq \mathbb{E}^{n-1}, f : F \to E, k : K \to E, S = S \subseteq \mathbb{E}^{n-1}, k \in K, \text{ and } S \cap K = \emptyset. \text{ Then } \\
  f \circ k \leq f|_S \circ k
\]

\[
  (1) f|_S \circ k \leq f \circ k \leq f|_S \circ k
\]

\[
  (2) f|_S \circ k = f|_S \circ k
\]

\[
  (3) a \circ k \circ \alpha = f|_S \circ k
\]

\[
  \text{then } a \leq f|_S \circ k \Rightarrow a = f|_S \circ k
\]

\[
  (4) a \circ k \circ \alpha = f|_S \circ k
\]

\[
  \text{then } a \geq f|_S \circ k \Rightarrow a = f|_S \circ k
\]

V. Conclusions

Our analysis shows that under reasonable sampling conditions, reconstruction can be recovered perfectly from the samplings. A set contains its reconstruction by closing and is contained in its reconstruction by dilation, and indeed these are extremal bounding sets. That is, the largest set which downsamples to a given set is its reconstruction by dilation; the smallest is its reconstruction by closing.

These results are proven for the binary case and stated for grey-scale morphology. Morphological sampling thus provides reconstructions positioned only within some spatial tolerance, in contrast to sampling in signal processing which reconstructs only those frequencies bandwidth limited to the Nyquist frequency.

This work provides a basis for multi-resolution shape analysis in shape-based hierarchal structures, to combine the advantages of multi-resolution techniques with those of morphological operators on cellular arrays. Acknowledgement. We wish to thank Laura Simpkins and Steve Kenyon of Digital Signal Corporation for, respectively, carefully creating the scene to our specifications and measuring all its sizes and distances, and range mapping the scene with DSC's custom FM laser radar.

REFERENCES


Figure 1. A 5-level pyramid of a laser radar range image produced by pure sampling. The highest resolution image size is 256 by 256. A 2 pixel wide line segment and a 4 x 4 box are intentionally placed at the upper right and upper left of the image, respectively.

Figure 2. A 5-level pyramid of the same image as figure 1, produced by 3 x 3 box filtering and then sampling.

Figure 3. A 3-D plot illustrates the greyscale opening operation. (a) shows a 20 x 20 checkerboard image with checker size 10 x 10. The grey value of the bright checker is 100 and the grey value of the dark checker is 50. (b) shows the noisy image by adding a zero mean Gaussian noise of standard deviation 20 to (a). (c) shows the result of opening (b) by a brick of size 10 x 10.

Figure 4. A 5-level pyramid of the same image as figure 1, produced by, opened by a brick of 3 x 3 and then sampling to generate the next layer.

Figure 5. A 5-level pyramid of the same image as figure 1. In each level, the image is opened by a brick of 3 x 3 and then is sampled and reconstructed before it is down sampled to generate the next level.