Model-Based Morphology

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1. Introduction

Filtering by morphological operations is particularly suited for removal of clutter and noise objects which have been introduced into noiseless binary images. The morphological filtering is designed to exploit differences in the spatial nature (shape, size, orientation) of the objects (connected components) in the ideal noiseless images as compared to the noise/clutter objects.

Since the typical noise models (union, intersection set difference, etc.) for binary images are not additive, and the morphological processing is strongly nonlinear, optimal filtering results conventionally available for linear processing in the presence of additive noise are not directly applicable to morphological filtering of binary images.

In this paper we describe a morphological filtering analog to the classic Wiener filter. The discussion begins in Section 2 with a review of the Wiener filter and its extension to a Binary Wiener filter; in these the underlying model entails decomposing the signal and additive noise into spectral elements in terms of an orthogonal basis set. Classic Wiener optimal estimation weights the respective spectral elements in the noisy signal according to the expected values of signal and noise energy across the spectrum. Section 3 extracts the essence of the algebraic structure underlying the derivation of the Wiener filter, doing so in a way that retains the concepts of energy and spectral decomposition, but eliminates the assumptions of noise additivity, orthogonal bases, and even the concept of inner product. The stage is thus set for the subsequent morphological filtering results where those assumptions do not apply. Section 4 derives an optimal morphological filter for binary images composed of the union (not sum) of the signal and noise connected components. The spectral decomposition of signal and noise is in terms of an ordered basis of connected components where the ordering is based on the morphological opening operation. (Such a basis is, in a certain sense, a
"nested" collection of sets.) Thus the underlying model is based upon that ordered basis (which provides prototypes of signal and noise objects scattered throughout the binary image) and upon the morphological pattern spectrum derived from openings. Section 5 expands the results of Section 4 beyond allowing signal and noise objects to be taken from a single ordered morphological basis (e.g. an ordered set of discs). In Section 5, the collection of prototypes can include any number of ordered bases (e.g. an ordered set of discs, as well as an ordered set of squares, as well as several ordered sets of lines each at different orientations.)

2. The Wiener Filter

Let \( b_1, \ldots, b_n \) be an orthonormal basis. The model for the ideal random signal \( f \) is that \( f = \sum_{n=1}^{N} \alpha_n b_n \) where \( E[\alpha_n] = 0, V[\alpha_n] = \sigma_{\alpha_n}^2, \) and \( E[\alpha_m \alpha_n] = 0, m \neq n. \) The variances \( \sigma_{\alpha_n}^2 \) are taken to be known. The model for the random noise \( g \) is that \( g = \sum_{n=1}^{N} \beta_n b_n \) where \( E[\beta_n] = 0, V[\beta_n] = \sigma_{\beta_n}^2, \) and \( E[\beta_m \beta_n] = 0, m \neq n. \) Noise and signal are uncorrelated so that \( E[\alpha_n \beta_m] = 0. \)

The observed noisy signal is \( f + g = \sum_{n=1}^{N} (\alpha_n + \beta_n) b_n. \) The Wiener filtering problem is to determine weights \( w_1, \ldots, w_N \) to make the estimate \( \hat{f} \) of \( f, \hat{f} = \sum_{n=1}^{N} w_n (\alpha_n + \beta_n) b_n \) minimize \( E[\rho(f, \hat{f})], \) where \( \rho \) is a metric. In the case of Euclidean distance for the metric \( \rho, \) \( E[\rho(f, \hat{f})] = E[\|f - \hat{f}\|^2]. \)

Now,

\[
\|f - \hat{f}\|^2 = \sum_{n=1}^{N} \| \alpha_n b_n - \sum_{n=1}^{N} w_n (\alpha_n + \beta_n) b_n \|^2 \\
= \sum_{n=1}^{N} [w_n (\alpha_n + \beta_n) - \alpha_n]^2
\]

And

\[
E[\|f - \hat{f}\|^2] = \sum_{n=1}^{N} \left( E[(w_n (\alpha_n + \beta_n) - \alpha_n)^2] \right) \\
= \sum_{n=1}^{N} w_n^2 (\sigma_{\alpha_n}^2 + \sigma_{\beta_n}^2) - 2 w_n \sigma_{\alpha_n}^2 + \sigma_{\alpha_n}^2
\]

Hence, the minimizing weights are given by

\[
w_n = \frac{\sigma_{\alpha_n}^2}{\sigma_{\alpha_n}^2 + \sigma_{\beta_n}^2}.
\]

One can also define a binary Wiener filter, with weights restricted to 0 or 1. To determine the minimizing weights, we need just examine

\[
w_n^2 (\sigma_{\alpha_n}^2 + \sigma_{\beta_n}^2) - 2 w_n \sigma_{\alpha_n}^2 + \sigma_{\alpha_n}^2 = \begin{cases} \\
\sigma_{\beta_n}^2 & \text{if } w_n = 0 \\
\sigma_{\alpha_n}^2 & \text{if } w_n = 1
\end{cases}
\]
Hence, under the constraint that the \( w_n \in \{0, 1\} \), the minimizing weights are given by

\[
w_n = \begin{cases} 
0 & \text{if } \sigma_n^2 < \sigma^2_s \\
1 & \text{otherwise}
\end{cases}
\]

In this case the estimate \( \hat{f} = \sum_{n \in S} (\alpha_n + \beta_n) b_n \), where \( S = \{ n | w_n = 1 \} \). Thus the optimal binary Wiener filter retains that part of the spectrum where the expected signal energy exceeds the expected noise energy, and discards the rest.

3. Optimal Filtering in the Generalized Case

This section restates the binary Wiener filter results, retaining the classic algebraic structure under far less restrictive assumptions than those of Section 2. The new assumptions will in fact be consistent with the morphological filter we will develop in Section 4. The reader preferring an actual instance of the new results first and a generalization later is advised to reverse the order of reading this section and the next.

Specifically we now relax the assumptions of additive noise, vector norms, inner products, and orthonormal bases, replacing them with more general assumptions on the nature of noise inclusion, distance, energy, and spectral decomposition, and the relationships between them.

Let \( f \) be any binary image in a set \( B \) of binary images and \( \psi \) be a mapping (a spectral decomposition) taking \( f \) into the \( N \)-tuple \( f_1, \ldots, f_N \); that is \( \psi : B \rightarrow B^N \). (In the case of the Wiener filter, the \( N \)-tuple \( f_1, \ldots, f_N \) is \( (\alpha_1 b_1, \ldots, \alpha_N b_N) \). Here, we incorporate into each \( f_n \) both the scalar and the basis elements.) Let \( \psi^{-1} \) be the inverse mapping re-assembling \( f_1, \ldots, f_N \) back into \( f \); that is \( \psi^{-1} : B^N \rightarrow B \). The identity operator can be expressed as \( \psi \psi^{-1} \) and \( \psi^{-1} \psi \). For any two binary images \( f \) and \( g \) in \( B \) let there be defined a binary operation \( <> \) such that \( f <> g \) is also a binary image in \( B \). When \( g \) is the noise, \( f <> g \) corresponds to the observed noisy binary image. We require that \( <> \) and \( \psi \) satisfy the relationship

\[
\psi(f <> g) = (f_1 <> g_1, \ldots, f_N <> g_N).
\]

Let \( \rho \) be the function evaluating the closeness of one image to another. Hence \( \rho : B \times B \rightarrow [0, \infty) \). The function \( \rho \) must satisfy \( \rho(f, h) = \sum_{n=1}^{N} \rho(f_n, h_n) \) where \( \psi(f) = (f_1, \ldots, f_N) \) and \( \psi(h) = (h_1, \ldots, h_N) \).

For any binary image \( g \), we let \( \# \) represent the operator which quantifies the energy in the binary image \( g \); \( \#: B \rightarrow [0, \infty) \). The operator \( \# \) must satisfy \( \# g = \sum_{n=1}^{N} \# g_n \), for spectral decomposition \( \psi(g) = (g_1, \ldots, g_N) \). Finally, there is a relationship between \( \rho \) and \( \# \) :

\[
\rho(f <> g, f) = \# g.
\]

Let \( w_n \in \{0, 1\}, n = 1, \ldots, N \) be binary weights and let the filtered binary image have a representation \( (w_1(f_1 <> g_1), \ldots, w_N(f_N <> g_N)) \) where

\[
w_n(f_n <> g_n) = \begin{cases} 
    f_n <> g_n & \text{if } w_n = 1 \\
    \phi & \text{if } w_n = 0
\end{cases}
\]
and \( \phi \) is the binary image satisfying \( f < \phi = f \). The filtered binary image \( \hat{f} \) itself can then be written as \( \hat{f} = \psi^{-1}(w_1(f,<>g_1), \ldots, w_n(f,<>g_n)) \). In essence the effect of the filtering is obtained by nulling spectral content of the observed noisy binary image.

The optimal filter parameters \( w_n \) are chosen to minimize

\[
E[\rho(\hat{f}, f)] = E[\sum_{n=1}^{N} \rho(\hat{f}_n, f_n)] = \sum_{n=1}^{N} E[\rho(w_n(f,<>g_n), f_n)]
\]

\[
= \sum_{n=1}^{N} E[\begin{cases} \#g_n & \text{if } w_n = 1 \\ \#f_n & \text{if } w_n = 0 \end{cases}].
\]

Hence, the best value for \( w_n \) is given by

\[
w_n = \begin{cases} 0 & \text{if } E[\#f_n] < E[\#g_n] \\ 1 & \text{otherwise} \end{cases}
\]

This defines the index set \( S \) corresponding to the spectral content that will be included in the optimal filter output.

4. Optimal Binary Morphological Filter

To derive the optimal binary morphological filter, we need to define the ideal random image model, the random noise model, the relationship of the observed image to the ideal random image and random noise, the formulation of representation operator \( \psi \) from morphological operators, the energy measure \( \# \), and the closeness measure \( \rho \). We begin with the representation operator \( \psi \).

The representation operator \( \psi \) will be defined in terms of the opening morphological pattern spectrum. To set up our definition for \( \psi \) in a way which relates to the ideal random image and noise models, we note that the opening operator has the following property: If \( A = \bigcup_{i=1}^{n} A_i \), where each \( A_i \) is a connected component of \( A \), and \( K \) is a connected structuring element, then \( A \circ K = (\bigcup_{i=1}^{n} A_i) \circ K = \bigcup_{i=1}^{n} (A_i \circ K) \).

This property, that the opening of a union of connected components is the union of the openings, will be essential throughout our development. It is this property which motivates the following definition: Two sets \( A \) and \( B \) are said to not interfere with one another if and only if \( X \), a connected component of \( A \cup B \), implies that \( X \) is a connected component of \( A \) or of \( B \) but not both. It immediately follows that if \( A \) and \( B \) do not interfere with one another and \( K \) is a connected structuring element, then \( (A \cup B) \circ K = (A \circ K) \cup (B \circ K) \).

The morphological pattern spectrum operator \( \psi \) will be defined in terms of a set of openings. This set of openings will be based on the structuring elements in a naturally ordered morphological basis. We define a collection \( \kappa \) of structuring elements to be a morphological basis if and only if \( K \in \kappa \) implies \( K \) is connected and \( K, L \in \kappa \) implies \( K \circ L = K \) or \( K \circ L = \phi \). A morphological basis \( \kappa = \{K(1), \ldots, K(M)\} \) is naturally ordered if and only if

\[
K(i) \circ K(j) = \begin{cases} K(i) & j \leq i \\ \phi & j > i \end{cases}
\]
A simple example of an ordered morphological basis is a set of squares of increasing size.

Now we can define the operator \( \psi \) which produces a morphological pattern spectrum with respect to a naturally ordered morphological basis \( \mathcal{K} = \{K(1), \ldots, K(M)\} \). \( \psi \) is defined by \( \psi(A) = (A_0, \ldots, A_M) \) where \( A_m = A \circ K(m) - A \circ K(m+1) \) for \( m = 0, \ldots, M-1, A_M = A \circ K(M) \), and \( K(0) = 0 \). \( A_m \) is that part of \( A \) which is open under \( K(m) \) but not open under \( K(m+1) \), except for \( A_M \) which is the remainder open under \( K(M) \). It follows from this definition that for \( i \neq j \), \( A_i \cap A_j = \phi \). This happens because

\[
A_i \cap A_j = [A \circ K(i) - A \circ K(i+1)] \cap [A \circ K(j) - A \circ K(j+1)]
\]

\[
= [A \circ K(i) \cap A \circ K(j)] \cap [A \circ K(i+1) \cup A \circ K(j+1)]
\]

\[
= [A \circ K(\max(i,j))] \cap [A \circ K(\min(i+1,j+1))]^c
\]

\[
= \phi \text{ since } \max(i,j) \geq \min(i+1,j+1) \text{ for any } i \neq j.
\]

Note that the above proof requires a slight modification for \( i \) or \( j = M \), because of the special definition of \( A_M \).

Next we discuss the spatial random process generation mechanism which produces binary image realizations. A spatial random process producing a set \( A \) is a non-interfering spatial Poisson process with respect to an ordered morphologic basis \( \mathcal{K} \) if and only if:

- For some \( Z \), a Poisson distributed random number (with Poisson density parameter \( \lambda_A \)), which is the total connected component count of a binary image realization \( A \);
- For some multinomial distributed numbers \( L_1, \ldots, L_M \) with \( \sum_{m=1}^{M} L_m = Z \) (with respective multinomial probabilities \( p_1, \ldots, p_M \)), which split the \( Z \) connected components into \( M \) subsets containing objects of the same type;
- For some randomly chosen translations \( x_{mj} \), \( m = 1, \ldots, M; j = 1, \ldots, L_m \);
- \( A = \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \), where the translated structuring elements do not interfere, i.e.,

\[
K(i)_{x_{ij}} \cap K(m)_{x_{mn}} = \begin{cases} 
K(i)_{x_{ij}} & \text{if } i = m \text{ and } j = n \\
\phi & \text{otherwise.}
\end{cases}
\]

From this definition of a non-interfering random process, it follows that

\[
(1) \quad A \circ K(\lambda) = \left( \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} K(m)_{x_{mj}} \right) \circ K(\lambda)
\]

\[
= \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} [K(m)_{x_{mj}} \circ K(\lambda)]
\]

\[
= \bigcup_{m=1}^{M} \bigcup_{j=1}^{L_m} K(m)_{x_{mj}}
\]

\[
(2) \quad \text{If } \psi(A) = (A_0, \ldots, A_M) \text{ then } A_m = \bigcup_{j=1}^{L_m} K(m)_{x_{mj}}, \text{ for } m = 1, \ldots, M
\]

\[
A_0 = \phi
\]
Conclusions (1) and (2) are interpreted as follows: If \( A \) is opened by the \( \lambda \)th basis structuring element, all components originating from "smaller" (lower-numbered) basis structuring elements are removed; the morphological pattern spectrum of \( A \) (with respect to the basis from which it was built) sorts \( A \) according to the index number of the underlying basis structuring elements, and leaves nothing out; the \( A_0 \) bin is empty because it receives only the part of \( A \) that is not open under \( K(1) \), but \( A \) was constructed with no components "smaller" than \( K(1) \).

We consider both the ideal random image and the noise image to be generated by non-interfering random spatial processes. The observed noisy image is the union of the ideal image with a noise/clutter image. This motivates a definition of non-interfering spatial processes which here plays the role of the zero correlation between the coefficients of the image process and the coefficients of the noise process in the Wiener filter case. A random process generating realization \( D \) and a random process generating realization \( E \) are said to be non-interfering random processes if and only if \( D \) and \( E \) are always non-interfering sets for each realization.

We can now define an observed noisy image. Let \( A \) be a realization of a non-interfering spatial process (with respect to an ordered morphological basis \( \kappa \)) producing images of interest and let \( N \) be a realization of a non-interfering spatial process (with respect to the same \( \kappa \)) producing noise/clutter. We suppose that these processes do not interfere with one another. The observed noisy realization is defined as \( A \cup N \). Let \( \psi(A \cup N) = (B_1, \ldots, B_M) \).

Then \( B_m = (A \cup N) \circ K(m) - (A \cup N) \circ K(m + 1) \), with \( B_M = (A \cup N) \circ K(M) \). Because the processes do not interfere with one another,

\[
B_m = [A \circ K(m) \cup N \circ K(m)] - [A \circ K(m + 1) \cup N \circ K(m + 1)]
= [A \circ K(m) - A \circ K(m + 1)] \cup [N \circ K(m) - N \circ K(m + 1)]
= A_m \cup N_m, \text{ where } \psi(A) = (A_0, \ldots, A_M) \text{ and } \psi(N) = (N_0, \ldots, N_M).
\]

Thus we have just seen that \( \psi(A \cup N) = (A_1 \cup N_1, \ldots, A_M \cup N_M) \), with \( A_0 = N_0 = \phi \) as before.

The filtered image \( \hat{A} \) will be based on selecting the most appropriate components from the morphological pattern spectrum of \( A \cup N \). So we estimate \( A \) by \( \hat{A} \) where

\[
\hat{A} = \bigcup_{m \in S} (A_m \cup N_m) \text{ or } \hat{A} = \bigcup_{m \in S} B_m.
\]

Thus by choosing the form of the estimation analogously to that of the binary Wiener filter, the estimation problem becomes one of choosing an appropriate index set \( S \).

To determine \( S \), we must first state our error criterion. For any two sets \( A \) and \( \hat{A} \), we define the closeness (non-overlap) of \( A \) to \( \hat{A} \) by \( \rho(A, \hat{A}) = \#(A - \hat{A} \cup \hat{A} - A) \) where \# is the set counting measure (pixel count, area). Our error criterion is then

\[
E[\rho(A, \hat{A})] = E\left[\#(A - \hat{A} \cup \hat{A} - A)\right].
\]
To see how to choose $S$ to minimize $E\left[ \#(A - \hat{A} \cup \hat{A} - A) \right]$, first note that

$$A - \hat{A} = \bigcup_{m=1}^{M} A_m - \bigcup_{m \in S} (A_m \cup N_m) = \bigcup_{m=1}^{M} A_m - \bigcup_{m \notin S} N_m,$$

$$\hat{A} - A = \bigcup_{m \in S} A_m \cup N_m - \bigcup_{m=1}^{M} A_m = \bigcup_{m \in S} N_m.$$

Hence,

$$\rho(A, \hat{A}) = \#(A - \hat{A} \cup \hat{A} - A)$$

$$= \#(A - \hat{A}) + \#(\hat{A} - A)$$

$$= \# \bigcup_{m \notin S} A_m + \# \bigcup_{m \in S} N_m$$

$$= \sum_{m=1}^{M} \#A_m + \sum_{m \in S} \#N_m.$$

The two summations above are respectively the area of the ideal image left out, plus the noise and clutter area left in. The individual terms decompose that area by spectral content.

Now, since each spectral component is built of translates of the same basis structuring elements,

$$\#A_m = \# \bigcup_{j=1}^{L_m} K(m)_{x,m,j}$$

$$= \sum_{j=1}^{L_m} \#K(m)_{x,m,j} = L_m \#K(m)$$

so that $E[\#A_m] = \#K(m)p_m\lambda_A\mathcal{A}$ where $p_m$ is the multinomial probability for the ideal image process, $\lambda_A$ is the Poisson density parameter of the ideal image process, and $\mathcal{A}$ is the area of the image spatial domain. Likewise, $E[\#N_m] = \#K(m)q_m\lambda_N\mathcal{A}$, where $q_m$ is the multinomial probability for the noise process and $\lambda_N$ is the Poisson density parameter of the noise process.

To determine the index set $S$, we then have

$$E \left[ \#(A - \hat{A} \cup \hat{A} - A) \right] = E \left[ \sum_{m=1}^{M} \left\{ \#A_m + \#N_m \right\} m \notin S \right]$$

$$= \sum_{m=1}^{M} \left\{ E[\#A_m] + E[\#N_m] \right\} m \notin S.$$

Hence, the best $S$ is defined by $S = \{m | E[\#N_m] < E[\#A_m]\}$, or equivalently for the statistical assumptions made, $S = \{m | q_m\lambda_N < p_m\lambda_A\}$. A spectral component is retained according to the relative expectations of that component's "leave-out" of ideal image vs. "leave-in" of noise and clutter.
5. Extension to Generalized Openings

The results we have just obtained can be extended to where the opening operation is changed to a generalized opening operation. Recall that in the previous section, each basic structuring element was just a set \( K \). In the generalized opening operation, each basic structuring element is a collection \( Q \) of sets. The generalized opening of an image \( I \) with \( Q \) is then defined by:

\[
I \circ Q = \bigcup_{L' \in Q} I \circ L'.
\]

This generalization is important because of the way it extends the underlying signal and noise spatial random process generation mechanism. For example, if the structuring elements were all line segments, the structuring element collection \( Q \) could consist of multiple orientation of line segments of the same length. The corresponding spatial random process would place non-interfering line segments at different orientations on the image. Or, the spatial random process could place non-interfering line segments, disks, or squares, on the image. For each size, the corresponding structuring element collection could be: line segments of the given size at a variety of orientations, a disk of the given size, and a square of the given size.

To see how the generalized opening can be used, we illustrate the case for which each structuring element collection contains exactly two structuring elements. Let \( \mathcal{K} = \{K(1), \ldots, K(M)\} \) and \( \mathcal{J} = \{J(1), \ldots, J(M)\} \) be naturally ordered morphological base. Define the collection \( Q \) by \( Q = \{Q(1), \ldots, Q(M)\} \) where \( Q(m) = \{K(m), J(m)\}, m = 1, \ldots, M \). To make the ordering of the collection \( \mathcal{K} \) and the collection \( \mathcal{J} \) compatible, we require that \( K(i) \circ J(j) = J(i) \circ K(j) = \phi \) for \( j > i \).

Now using the generalized opening operator, consider

\[
K(i) \circ Q(j) = K(i) \circ K(j) \cup K(i) \circ J(j)
\]

\[
= \begin{cases} 
K(i) & i \geq j \\
\phi & \text{otherwise}
\end{cases}
\]

Likewise,

\[
J(i) \circ Q(j) = J(i) \circ K(j) \cup J(i) \circ J(j)
\]

\[
= \begin{cases} 
J(i) & i \geq j \\
\phi & \text{otherwise}
\end{cases}
\]

Suppose that a realization \( A \) for a non-interfering process can be written as

\[
A = \bigcup_{m=1}^{M} \bigcup_{i=1}^{L_x^m} K(m)_{x_m} \bigcup_{m=1}^{M} \bigcup_{i=1}^{L_y^m} J(m)_{y_m},
\]
where the sets in the collection \( \{K(m)_{x_{m_i}}, J(m)_{y_{m_j}} : i = 1, \ldots, L^K_m, j = 1, \ldots, L^J_m \} \) are naturally non-interfering. Then

\[
A \circ Q(\lambda) = \left[ \bigcup_{m=1}^{M} \bigcup_{j=1}^{L^K_j} K(m)_{x_{m_i}} \bigcup_{m=1}^{M} \bigcup_{j=1}^{L^J_j} J(m)_{y_{m_j}} \right] \circ Q(\lambda)
\]

\[
= \bigcup_{m=1}^{M} \bigcup_{j=1}^{L^K_j} [K(m)_{x_{m_i}} \circ Q(\lambda)] \bigcup_{m=1}^{M} \bigcup_{j=1}^{L^J_j} [J(m)_{y_{m_j}} \circ Q(\lambda)]
\]

\[
= \bigcup_{m=1}^{M} \bigcup_{j=1}^{L^K_j} K(m)_{x_{m_i}} \bigcup_{m=1}^{M} \bigcup_{j=1}^{L^J_j} J(m)_{y_{m_j}}
\]

\[
A_m = A \circ Q(m) - A \circ Q(m+1)
\]

\[
= \bigcup_{m=1}^{M} \bigcup_{j=1}^{L^K_j} K(n)_{x_{n_i}} \bigcup_{m=1}^{M} \bigcup_{j=1}^{L^J_j} J(n)_{y_{n_j}}
\]

\[
- \bigcup_{m=m+1}^{M} \bigcup_{j=1}^{L^K_j} K(n)_{x_{n_i}} \bigcup_{m=m+1}^{M} \bigcup_{j=1}^{L^J_j} J(n)_{y_{n_j}}
\]

\[
= \bigcup_{j=1}^{L^K_j} K(m)_{x_{m_i}} \bigcup_{j=1}^{L^J_j} J(m)_{y_{m_j}}
\]

From this it is clear that the representation operator \( \psi \) based on \( Q \) has an inverse and \( A = \bigcup_{m=1}^{M} A_m \). Furthermore, \( A_i \cup A_j = \phi \) and \( \# A = \sum_{m=1}^{M} \# A_m \). This fulfills the required conditions described in Section 3. Furthermore, results for \( Q \) containing collections of pairs of structuring elements are immediately generalizable to collections having any number of structuring elements.

6. Conclusion

For the problem of filtering corrupted binary images of the form \( A \cup N \), we have chosen an appropriate spectral decomposition, as well as distance and energy measures resulting in an appropriate measure of estimation error. Based upon these choices (which are quite different from the analogous choices for the additive noise/linear filter problem, and which eliminate the requirement for orthogonality or an inner product space) we have derived optimal filtering results analogous to conventional Weiner filtering results based on image and noise energy contents in each spectral bin.

The assumptions on the image and noise models in order for the results to be valid are presently fairly strong. The image and noise connected components are modeled as translated copies of objects from a single ordered morphological basis set (Section 4) or a collection of such basis sets (Section 5). In addition there is a non-interference (non-overlap) condition so that all objects remain distinct and no objects are created that fail to arise directly from basis sets.
In order to better handle irregular or ill-defined noise sets, as well as ideal (noise free) images comprised of families of objects for which no simple ordered morphological basis is obvious, we are working on extending our results to instances where the assumptions on image and noise objects are relaxed. In particular, extension to the case where the objects are in some sense well-sorted by one or more morphological bases, is being sought in derivations and experiments.

In addition, because of the algebraic structure laid out in Section 3, the derived results appear to be generalizable beyond opening pattern spectra. Analogous results can be shown for the closing pattern spectrum. Indeed the opening or closing does not even have to be the morphological opening or closing. These results extend to any case in which there is an operator which is extensive, increasing, idempotent, or when the operator is anti-extensive, increasing, and idempotent. For example the operator taking $A$ to $A \ominus K \cap A$ for any structuring element $K$ that is symmetric is anti-extensive, increasing, and idempotent. Likewise, any operator taking $A$ to $(A \oplus K) \cup A$ is extensive, increasing, and idempotent.

7. Acknowledgement

This work was supported by the Office of Naval Research under its ARL Program (Grant N0014-90-J-1369).