Mathematical Morphology and the Morphological Sampling Theorem

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ABSTRACT

For the purpose of object or defect identification required in vision applications in manufacturing, the operations of mathematical morphology are perhaps more useful than the convolution operations. This is because mathematical morphology provides a natural algebraic theory for working with shape. The morphological sampling theorem described in this paper states how a digital image must be morphologically filtered before sampling in order to preserve relevant information after sampling. The sampling theorem indicates what precision an appropriately morphologically filtered image can be reconstructed after sampling, and it specifies the relationship between morphological processing before sampling and the more computationally efficient scheme of morphologically operating on the sampled image. Thus, the morphological sampling theorem provides a sound basis for recognizing and extracting shape information in a computationally efficient, multi-resolution pyramid processing approach.

1. Overview

Section 2 briefly describes the vision group at the University of Washington. Sections 3 and 4 review the basic morphological operations and the algebra defined by them. Sections 5 through 7 develop the binary and grayscale morphological sampling theorem. Section 8 briefly discusses other vision research being done at the University of Washington and where we expect to be going.

2. UW Vision Group

The University of Washington has assembled a vision group whose senior researchers are internationally known and recognized for their contributions in image analysis and computer vision. The primary members of the group are Robert M. Haralick, Linda G. Shapiro, Steven L. Tanimoto, Kenneth Sloan, John Palmer, Arun Somani and Mani Soma. Professor Robert Haralick holds the Boeing Clairmont Egtvedt Chair in Electrical Engineering and has an adjunct appointment in Computer Science. He has published several hundred papers on a variety of vision topics including texture, facet model, consistent labeling, and most recently on mathematical morphology. His texture papers showed the utility of the gray tone co-occurrence matrix. The facet model papers showed that a variety of low-level feature extraction operations such as edge, line, and corner detection can be viewed as possessing a locally estimated underlying gray tone intensity surface of which the given image is a sampled noisy version. The consistent labeling papers recognized the variety of vision problems which are special cases of consistent labeling. These papers developed the general theory behind the relaxation and look ahead techniques which speed up the tree search.

Professor Haralick has served as head of the IEEE Computer Society Pattern Analysis and Machine Intelligence Technical Committee. He now serves as the head of the Computer Society Task Force on Artificial Intelligence. He is a Fellow of the IEEE for his contributions in image processing and computer vision. He serves on the Editorial Board of the IEEE Transactions on Pattern Analysis and Machine Intelligence. He is the computer vision area editor for Communications of the ACM. He also serves as an associate editor for Computer Vision, Graphics, and Image Processing as well as Pattern Recognition.

Professor Linda G. Shapiro holds an appointment in the Electrical Engineering Department in the Computer Engineering program, as well as an adjunct appointment in Computer Science. She is the Editor of Computer Vision, Graphics, and Image Processing, the journal which is the grandfather journal for archival quality papers on computer vision. Professor Shapiro has designed a language and associated recognition system for expressing the structural relationships among entities in an image. She has defined a structural representation for describing two-dimensional shapes and implemented an associated shape matching procedure. She has developed an inexact matching methodology which permits, in special cases, efficient comparison (polynomial time) of two structures which are not identical as well a method for organizing relational models into clusters of similar models so that an unknown object can be compared only against the cluster representatives. She has
worked on the design of INSIGHT, a dataflow language for expressing vision and AI algorithms in parallel architectures.

Professor Steven L. Tanimoto holds an appointment in the Computer Science Department (which has been acknowledged to be one of the top ten in the country) and an adjunct appointment in the Electrical Engineering Department. He is the Editor in Chief of the *IEEE Transactions on Pattern Analysis and Machine Intelligence* which has historically established itself as the premiere journal for vision research. Professor Tanimoto works in the area of parallel architecture and algorithms for machine vision. He has formulated a hierarchical architecture for high-speed vision. This architecture embodies several novel features. One of these is the use of a parallel–pyramidal interconnection network allowing both local and global image transforms to be computed rapidly. Another is hardware for a parallel operator for hierarchical cellular logic that can transform every cell in parallel according to all the values in its 14–point hierarchical neighborhood, in a single machine instruction. On the algorithms side, Professor Tanimoto has developed a fast Hough transform algorithm for the architecture; it uses a novel bottom–up clustering procedure that achieves $O(\log N)$ computation times. He has also developed several fast image search algorithms. At the meta–algorithm level, he has formulated a model for automatic algorithm development in the machine vision context; the model uses concepts from state-space as well as image processing.

Dr. Kenneth Sloan holds an appointment as Assistant Professor in the Computer Science Department. He has experience in computer graphics, vision, artificial intelligence, and networks. His dissertation at the University of Pennsylvania treated the problem of analyzing monocular, color views of natural outdoor scenes. At the University of Rochester, he worked on a broad range of vision problems as part of the DARPA Image Understanding program. He worked with Peter Selfridge on the use of adaptive low–level vision in the domain of aerial image understanding. While at the Architecture Machine Group at the Massachusetts Institute of Technology, he worked primarily in the domain of 3D reconstruction and display of (usually biological) surfaces determined from series of 2D slices. He has been at the University of Washington for over 3 years, during which time he has directed the establishment of the GRaphics and Artificial Intelligence Lab (GRAIL). His recent work on reconstruction and display of the human retina was featured on the cover of *Science*.

Dr. John Palmer holds an appointment as Assistant Professor in the Psychology Department. He has been conducting research in human vision and human performance. He has investigated the features mediating vision for spatial relations and motion. The research defined a general mathematical model of a feature and generated predictions that can be used to distinguish between alternative hypotheses of what features mediate a particular perceptual judgment. In experimental work, he tested alternative feature theories for localizing moving stimuli. The experiments demonstrated that motion contributes a unique feature to visual localization under certain temporal conditions and not under others. These experiments provided a rigorous confirmation of informal intuitions concerning perceived motion. The feature theory and experimental methodology developed in these studies can be readily generalized to other perceptual domains.

Dr. Palmer has also investigated various limitations on human performance. Several recent studies have measured the limits on human perception imposed by attention and memory. These studies have provided new quantifications of the capacity limits of human observers. In previous studies, he has searched for good predictors of skilled performance in reading and has studied what learning conditions allow for highly–skilled performance in laboratory categorization tasks.

Dr. Arun Somani holds an appointment as an Assistant Professor in Electrical Engineering. He has several years of experience in developing real time systems and does research in computer architecture, neural networks, parallel processing and fault tolerant computing. In his dissertation at McGill University he developed a parallel fault diagnosis algorithm for a multi–processing environment and a theory characterizing partial diagnosable systems. Earlier, he designed a VLSI architecture for performing various searches and maintaining parallel data dictionaries. He recently developed a parallel pipelined architecture for determining line of sight visibility for graphics application. His current research interest is in special purpose parallel computer architectures for high–performance and high reliability systems.

Dr. Mani Soma holds an appointment as an Assistant Professor in Electrical Engineering after receiving his Ph.D. degree from Stanford University in 1980 and working for two years at the General Electric Research and Development Center (Schenectady, New York). His research interests include the design, testing and reliability characterization of integrated circuits and systems. Since 1984 he has focused on the architectures and designs of application–specific integrated circuits (ASICs), where the emphasis is on high–performance digital–analog systems for applications in vision, image processing, and signal processing. Testability and reliability aspects of these circuits are emphasized and considered in performance tradeoffs.

3. Overview and Summary

Mathematical morphology provides an approach to the processing of digital images which is based on shape. Appropriately used, mathematical morphological operations
tend to simplify image data, preserving their essential shape characteristics and eliminating irrelevancies. As the identification of objects, object features, and assembly defects correlate directly with shape, it becomes apparent that the natural processing approach to deal with the manufacturing machine vision recognition process and the visually guided robot problem is mathematical morphology.

Machines which perform morphologic operations are not new. They are the essence of what cellular logic machines such as the Golay logic processor (Golay, 1969), Diff3 (Graham and Norgren, 1980), PICAP (Kruse, 1977), the Leitz Texture Analysis System TAS (Klein and Serra, 1977), the CLIP processor arrays (Duff, 1979), and the Delft Image Processor DIP (Gerritsen and Aardema, 1981) all do. A number of companies now manufacture industrial vision machines which incorporate video rate morphological operations. These companies include International Robomation Inc., Allen Bradley, 3M, Machine Vision International, Maitre, Synthetic Vision Systems, Vicom, Applied Intelligence Systems, Inc., and Leitz. Most of the real time VME and multibus machine vision boards developed by companies such as Datacube, Recognition Technology Inc., and Imaging Technology Inc. all support morphological operations.


Although the techniques are being used in the industrial world, the basis and theory of mathematical morphology tend to be (with the exception of the highly mathematical books by Matheron (1975) and Serra (1982) and the more readable chapter in the book by Dougherty and Giardina (1987)) not covered in the textbooks and, until recently, not covered in the journals which discuss image processing or computer vision.

Not only are the operations of mathematical morphology natural for shape processing, but morphological operations on images have relevance to the entire suite of conditioning, labeling, grouping, extracting and matching image processing operations. Thus from low-level to intermediate to high-level vision, morphological techniques are important. Indeed, many successful machine vision algorithms employed in industry on the factory floor, processing thousands of images per day in each application, are based on morphological techniques. Among the recent research papers on morphology are Crimmons and Brown (1985), Zhuang and Haralick (1985), Lee, Haralick, and Shapiro (1987), Haralick, Sternberg, and Zhuang (1987), and Maragos and Schafer (1987). The September 1986 issue of Computer Vision, Graphics, and Image Processing is devoted to morphology.

Many well-known relationships worked out in the classical context of the convolution operation have morphological analogs. In this paper, we introduce the digital morphological sampling theorem, which relates to morphology as the standard sampling theorem relates to signal processing and communications. The sampling theorem permits the development of a precise multiresolution approach to morphological processing.

Multiresolution techniques (Ahuja and Samy, 1984; Klinger, 1984; Meresereau and Speake, Tanimoto, 1982; Uhr, 1983) have been useful for at least two fundamental reasons: (1) the representation they provide naturally permits a computational mechanism to focus on objects or features likely to be at least a given specified size (Crowley, 1984; Miller and Stout; Rosenfeld, 1983; Witten, 1984), and (2) the computational mechanism can operate on only those resolution levels which just suffice for the detection and localization of objects or features of specified size while significantly reducing the number of operations performed (Burt, 1984; Dyer, 1982; Lougheed and McCubbrey, 1980).

The usual resolution hierarchy, called a pyramid, is produced by low pass filtering and then sampling to generate the next lower resolution level of the hierarchy. The basis for a morphological pyramid requires a morphological sampling theorem which explains how an appropriately morphologically filtered and sampled image relates to the unsampled image. It must explain what kinds of shapes are preserved and what kinds are suppressed or eliminated. It must explain the relationship between performing a less costly morphological filtering operation on the sampled image and performing the more costly equivalent morphological filtering operation on the original image. It is just these issues which we address in this paper.

The following results are shown to be true under reasonable morphological sampling conditions. Before sets are sampled, they must be morphologically simplified by an opening or a closing. Such sampled sets can be reconstructed in two ways, by either a closing or a dilation. In both reconstructions, the sampled reconstructed sets are equal to the sampled sets. For binary morphology a set contains its reconstruction by closing and is contained in its reconstruction by dilation; indeed, these are extremal bounding sets. That is, the largest set which downsamples to a given set is its reconstruction by dilation; the smallest is its reconstruction by closing. Furthermore, the distance from the maximal reconstruction to the minimal reconstruction is no more than the diameter of the reconstruction structuring element. Equivalent relations hold in the grayscale morphology. Morphological sampling thus provides reconstructions positioned only to within some spa-
tial tolerance which depends on the sampling interval. This spatial limitation contrasts with the sampling reconstruction process in signal processing from which only those frequencies below the Nyquist frequency can be reconstructed.

A number of relationships follow from the morphological sampling theorem. These relationships govern the commutivity between sampling and then performing morphological operations in the sampled domain versus first performing the morphological operations and then sampling. We find that sampling a minimal reconstruction which has been dilated is identical to dilating the sample set with a sampled structuring element. Sampling a maximal reconstruction which has been eroded is identical to eroding the sampled set with a sampled structuring element. These results establish bounds which can be used to determine the difference between morphological operations in the sampled domains and operations in the original domain followed by sampling.

All set morphological relationships are immediately generalizable to gray scale morphology via the umbra homomorphism theorems. For grayscale images, the bounds which the reconstruction establishes are bounds which are simultaneously grayscale and spatial.

4. The Morphological Operations

The language of mathematical morphology is that of set theory. Sets in mathematical morphology represent the shapes which are manifested on binary or gray tone images. The set of all the black pixels in a black and white image, (a binary image) constitutes a complete description of the binary image. Sets in Euclidean 2-space denote foreground regions in binary images. Sets in Euclidean 3-space may denote time varying binary imagery or static grayscale imagery as well as binary solids. Sets in higher dimensional spaces may incorporate additional image information, like color, or multiple perspective imagery. Mathematical morphological transformations apply to sets of any dimensions, those like Euclidean N-space, or those like its discrete or digitized equivalent, the set of N-tuples of integers, \( Z^N \). For the sake of simplicity we will refer to either of these sets as \( E^N \).

Those points in a set being morphologically transformed are considered as the selected set of points and those in the complement set are considered as not selected. Hence, morphology from this point of view is binary morphology. We begin our discussion with the binary morphological operations of dilation and erosion and then extend this discussion to gray scale morphology.

4.1 Dilation and Erosion

Dilation is the morphological transformation which combines two sets using vector addition of set elements. If \( A \) and \( B \) are sets in N-space \( (E^N) \) with elements \( a \) and \( b \) respectively, \( a = (a_1, ..., a_N) \) and \( b = (b_1, ..., b_N) \) being \( N \)-tuples of element coordinates, then the dilation of \( A \) by \( B \) is the set of all possible vector sums of pairs of elements, one coming from \( A \) and one coming from \( B \).

Let \( A \) and \( B \) be subsets of \( E^N \). The dilation of \( A \) by \( B \) is denoted by \( A \oplus B \) and is defined by

\[
A \oplus B = \{ c \in E^N | c = a + b \text{ for some } a \in A \text{ and } b \in B \}
\]

Erosion is the morphological dual to dilation. It is the morphological transformation which combines two sets using the vector subtraction of set elements. If \( A \) and \( B \) are sets in Euclidean N-space, then erosion of \( A \) by \( B \) is the set of all elements \( x \) for which \( x + b \in A \) for every \( b \in B \). Some image processing people use the name shrink or reduce for erosion.

The erosion of \( A \) by \( B \) is denoted by \( A \ominus B \) and is defined by

\[
A \ominus B = \{ x \in E^N | x + b \in A \text{ for every } b \in B \}.
\]

For any set \( A \subseteq E^N \) and \( x \in E^N \), let \( A_x \) denote the translation of \( A \) by \( x \);

\[
A_x = \{ y | \text{ for some } a \in A, y = a + x \}.
\]

For any set \( A \subseteq E^N \), let \( \hat{A} \) denote the reflection of \( A \) about the origin;

\[
\hat{A} = \{ x | \text{ for some } a \in A, x = -a \}.
\]

Relationships satisfied by dilation and erosion include the following:

\[
A \oplus B = B \oplus A
\]

\[
(A \oplus B) \oplus C = A \oplus (B \oplus C)
\]

\[
(A \ominus B) \ominus C = A \ominus (B \ominus C)
\]

\[
(A \cup B) \ominus C = (A \ominus C) \cup (B \ominus C)
\]

\[
(A \cap B) \ominus C = (A \ominus C) \cap (B \ominus C)
\]

\[
A \ominus B = \bigcup_{b \in B} A_b
\]

\[
A \ominus B = \bigcup_{A \ominus b} A_{\ominus B}
\]

\[
A \subseteq B \Rightarrow A \oplus C \subseteq B \oplus C
\]

\[
A \subseteq B \Rightarrow A \ominus C \subseteq B \ominus C
\]

\[
(A \cap B) \ominus C \subseteq (A \ominus C) \cap (B \ominus C)
\]

\[
(A \ominus B) \ominus C = (A \ominus C) \cup (B \ominus C)
\]

\[
(A \oplus B) \ominus C = (A \ominus C) \cup (B \ominus C)
\]

\[
A \oplus (B \cup C) = (A \oplus B) \cap (A \oplus C)
\]

4.2 Opening and Closing

In practice, dilations and erosions are usually employed in pairs, either dilation of an image followed by the erosion of the dilated result, or image erosion followed by dilation. In either case, the result of iteratively applied dilations and erosions is an elimination of specific image detail smaller than the structuring element without the global geometric distortion of unsuppressed features. For example, opening
an image with a disk structuring element smooths the contour, breaks narrow isthmuses, and eliminates small islands and sharp peaks or capes. Closing an image with a disk structuring element smooths the contours, fuses narrow breaks and long thin gulls, eliminates small holes, and fills gaps on the contours.

Of particular significance is the fact that image transformations employing iteratively applied dilations and erosions are idempotent, that is, their reapplication effects no further changes to the previously transformed result. The practical importance of idempotent transformations is that they comprise complete and closed stages of image analysis algorithms because shapes can be naturally described in terms of under what structuring elements they can be opened or can be closed and yet remain the same. Their functionality corresponds closely to the specification of a signal by its bandwidth. Morphologically filtering an image by an opening or closing operation corresponds to the ideal non-realizable bandpass filters of conventional linear filtering. Once an image is ideal bandpass filtered, further ideal bandpass filtering does not alter the result.

These properties motivate the importance of opening and closing, concepts first studied by Matheron (1967, 1975) who was interested in axiomatizing the concept of size. Both Matheron’s (1975) definitions and Serra’s (1982) definitions for opening and closing are identical to the ones given here, but their formulas appear different because they use the symbol $\ominus$ to mean Minkowski subtraction rather than erosion.

The morphological filtering operations of opening and closing are made up of dilation and erosion performed in different orders. The opening of $A$ by $B$ is defined by

$$A \circ B = (A \ominus B) \ominus B.$$  

The closing of $A$ by $B$ is defined by

$$A \bullet B = (A \oplus B) \ominus B.$$  

Opening and closing satisfy the following basic relationships:

$$(A \circ B) \circ B = A \circ B$$  

$$(A \bullet B) \bullet B = A \bullet B$$  

$$A \circ B \subseteq A$$  

$$A \subseteq (A \circ B)$$  

$$B \subseteq (A \bullet B)$$  

$$(A \circ B) \subseteq A \circ C$$  

$$(A \bullet B) \subseteq A \bullet C$$  

$$(A \circ B) \circ C = A \circ (B \circ C)$$  

$$(A \bullet B) \bullet C = A \bullet (B \bullet C)$$  

$$A \circ (B \ominus C) = (A \ominus B) \ominus (A \ominus C)$$  

$$A \bullet (B \ominus C) = (A \ominus B) \ominus (A \ominus C)$$  

$$(A \circ B) \circ (A \ominus B) = A \circ (B \ominus C)$$  

$$(A \bullet B) \bullet (A \ominus B) = A \bullet (B \ominus C)$$  

$$C \ominus (A \circ B) = C \ominus (A \bullet B)$$  

$$C \ominus (A \ominus B) = C \ominus (A \ominus B)$$  

$$C \ominus (A \ominus B) = C \ominus (A \ominus B)$$  

$$C \ominus (A \ominus B) = C \ominus (A \ominus B)$$  

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$$C \ominus (A \ominus B) = C \ominus (A \ominus B)$$  

The reason that openings and closings deal directly with shape properties is apparent from the following representation theorem for openings.

$$A \circ B = \{x \mid \text{for some } y, x \in B \cap A \}.$$  

A opened by $B$ contains only those points of $A$ which can be covered by some translation $B_y$ which is, in turn, entirely contained inside $A$. Thus $x$ is a member of the opening if it lies in some area inside $A$ which entirely contains a translated copy of the shape $B$. In this sense, $A$ opened by $B$ is the set of all points of $A$ which can participate in areas of $A$ which match $B$. If $B$ is a disk of diameter $d$, for example, then $A \circ B$ would be that part of $A$ which is no place is narrower than $d$.

The duality relationship $(A \circ B)^c = A^c \bullet B$ between opening and closing implies a corresponding representation theorem for closing.

$$A \bullet B = \{x \mid x \in B \cap A \neq \emptyset \}.$$  

A closed by $B$ consists of all those points $x$ for which $x$ being covered by some translation $B_y$ implies that $B_y \cap A \neq \emptyset$ or intersects some part of $A$. A more extensive discussion of these relationships can be found in Haralick, Sternberg, and Zhuang (1987).

4.3 Gray Scale Morphology


We will develop the extension in the following way. First we introduce the concept of the top surface of a set and the related concept of the umbra of a surface. Then gray scale dilation will be defined as the surface of the dilation of the umbra. From this definition we will prove the representation which indicates that gray scale dilation can be computed in terms of a maximum operation on a set of sums. A similar plan is followed for erosion which can be evaluated in terms of a minimum operation on a set of differences.
of each other. Then we illustrate how the umbra operation is a homomorphism from the gray scale morphology to the binary morphology. Having the homomorphism in hand, all the interesting relationships follow by appropriately unwrapping and wrapping the involved sets or functions.

4.3.1 Grayscale Dilation and Erosion

We begin with the concepts of surface of a set and the umbra of a surface. Suppose a set $A$ in Euclidean $N$-space is given. We adopt the convention that the first (N-1) coordinates of the N-tuples of $A$ constitute the spatial domain of $A$ and the $N^{th}$ coordinate is for the surface. For grayscale imagery, $N = 3$. The top or top surface of $A$ is a function defined on the projection of $A$ onto its first (N-1) coordinates. For each (N-1)-tuple $x$ the top surface of $A$ at $x$ is the highest value $y$ such that the N-tuple $(x, y) \in A$. This is illustrated in Figure 1. If the space we work in is Euclidean, we can express this using the concept of supremum. If the space is discrete, we use the more familiar concept of maximum. Since we have suppressed the underlying space in what follows, we use maximum throughout. The careful reader will want to translate maximum to supremum under the appropriate circumstances.

Let $A \subseteq E^N$ and $F = \{ x \in E^{N-1} \mid \text{for some } y \in E, (x, y) \in A \}$. The top or top surface of $A$, denoted by $T[A]: F \rightarrow E$, is defined by

$$T[A](x) = \max \{ y \mid (x, y) \in A \}$$

Figure 1 illustrates the concept of top or top surface of a set.

A set $A \subseteq E^{N-1} \times E$ is an umbra if and only if $(x, y) \in A$ implies that $(z, z) \in A$ for every $z \leq y$.

For any function $f$ defined on some subset $F$ of Euclidean $(N-1)$-space the umbra of $f$ is a set consisting of the surface $f$ and everything below the surface. Let $F \subseteq E^{N-1}$ and $f : F \rightarrow E$. The umbra of $f$, denoted by $U[f]$, $U[f] \subseteq F \times E$, is defined by

$$U[f] = \{(x, y) \in F \times E \mid y \leq f(x)\}$$

Obviously, the umbra of $f$ is an umbra.

Example This illustrates a discretized one-dimensional function $f$ defined on a domain consisting of seven successive column positions and a finite portion of its umbra which lies on or below the function $f$. The actual umbra has infinite extent below $f$. Having defined the operations of taking a top surface of a set and the umbra of a surface, we can define grayscale dilation. The gray scale dilation of two functions is defined as the surface of the dilation of their umbras.

Let $F, K \subseteq E^{N-1}$ and $f : F \rightarrow E$ and $k : K \rightarrow E$. The dilation of $f$ by $k$ is denoted by $f \oplus k$, $f \oplus k : F \oplus K \rightarrow E$, and is defined by

$$f \oplus k = T[U[f] \oplus U[k]]$$

Example This illustrates a second discretized one-dimensional function $k$ defined on a domain consisting of three successive column positions and a finite portion of its umbra which lies on or below the function $k$. The dilation of the umbras of $f$ (from the previous example) and $k$ are shown and the surface of the dilation of the umbras of $f$ and $k$ are shown.
The definition of grayscale dilation tells us conceptually how to compute the gray scale dilation, but this conceptual way is not a reasonable way to compute it in hardware. The following proposition establishes that grayscale dilation can be accomplished by taking the maximum of a set of sums. Hence, grayscale dilation has the same complexity as convolution. However, instead of doing the summation of products as in convolution, a maximum of sums is performed.

**Proposition** Let $f : F \to E$ and $k : K \to E$. Then $f \circ k : F \circ K \to E$ can be computed by

$$(f \circ k)(z) = \max_{z \geq k} \{f(x - z) + k(z)\}$$

**Proof** Suppose $z = (f \circ k)(x)$. Then $z = T[U[f] \circ U[k]](x)$. By definition of surface,

$$z = \max(y \mid (x, y) \in U[f] \circ U[k]).$$

By definition of dilation,

$$z = \max \{a + b \mid \text{for some } u \in K \text{ satisfying } z - u \in F, \ (z - u, a) \in U[f] \text{ and } (u, b) \in U[k]\}$$

By definition of umbrella, the largest $a$ such that $(x - u, a) \in U[f]$ is $a = f(x - u)$. Likewise, the largest $b$ such that $(u, b) \in U[k]$ is $b = k(u)$. Hence

$$z = \max_{(x - u) \in F} \{f(x - u) + k(u)\}$$

The definition for grayscale erosion proceeds in a similar way to the definition of grayscale dilation. The grayscale erosion of one function by another is the surface of the binary erosions of the umbrella of one with the umbrella of the other.

Let $F \subseteq E^{n-1}$ and $K \subseteq E^{n-1}$. Let $f : F \to E$ and $k : K \to E$. The erosion of $f$ by $k$ is denoted by $f \circ k$, $f \circ k : F \circ K \to E$, and is defined by

$$f \circ k = T[U[f] \circ U[k]].$$

**Example** Using the same function $f$ and $k$ of the previous example, illustrated here is the erosion of $f$ by $k$ by taking the surface of the erosion of the umbrella of $f$ by the umbrella of $k$.

Evaluating a grayscale erosion is accomplished by taking the minimum of a set of differences. Hence its complexity is the same as dilation. Its form is like correlation with the summation of correlation replaced by the minimum operation and the product of correlation replaced by a subtraction operation. If the underlying space is Euclidean, substitute infimum for minimum.

**Proposition** Let $f : F \to E$ and $k : K \to E$. Then $f \circ k : F \circ K \to E$ can be computed by $(f \circ k)(x) = \min_{u \in A} (f(x + u) - k(u))$

**Proof** Suppose $z = (f \circ k)(x)$. Then $z = T[U[f] \circ U[k]](x)$. By definition of surface, $z = \max(y \mid (x, y) \in U[f] \circ U[k])$. By definition of erosion

$$z = \max\{y \mid \text{for every } (u, v) \in U[k], \\text{there exists } z \text{ such that } (x, y) + (u, v) \in U[f]\}$$

By definition of umbra,

$$z = \max\{y \mid \text{for every } u \in K, \ (x - u, y) \in U[f], \ y + v \leq f(x + u)\}$$

But $y \leq f(x + u) - v$ for every $v \leq k(u)$ implies $y \leq f(x + u) - k(u)$. Hence

$$z = \max\{y \mid \text{for every } u \in K, \ y \leq f(x + u) - k(u)\}$$

But $y \leq f(x + u) - k(u)$ for every $u \in K$ implies

$$y \leq \min_{u \in K} [f(x + u) - k(u)]$$

Now,

$$z = \max\{y \mid y \leq \min_{u \in A} [f(x + u) - k(u)]\}$$

The basic relationship between the surface and umbrella operations is that they are, in a certain sense, inverses of each other. More precisely, the surface operation will always undo the umbrella operation. That is, the surface operation is a left inverse to the umbrella operation. However the umbrella operation is not an inverse to the surface operation. Without any constraints on the set $A$, the strongest statement which can be made is that the umbrella of the surface of $A$ contains $A$. When the set $A$ is an umbrella, then the umbrella of the surface of $A$ is itself $A$. In this case the umbrella operation is an inverse to the surface operation.

Having established that the surface operation is always an inverse to the umbrella operation and that the umbrella operation is the inverse to the surface operation when the set being operated on itself is an umbrella, we only need to note that the dilation of one umbrella by another is an umbrella and that the erosion of one umbrella by another is also an umbrella and we are ready to develop the umbrella homomorphism theorem.

The umbrella homomorphism theorem states that the operation of taking an umbrella is a homomorphism from the
gray scale morphology to the binary morphology.

**Umbral Homomorphism Theorem:** Let \( F, K \subseteq \mathbb{E}^{n-1} \) and \( f : F \to E \) and \( k : K \to E \). Then

\[
\begin{align*}
(1) \quad U[f \circ k] &= U[f] \circ U[k] \\
(2) \quad U[f \circ k] &= U[f] \circ U[k]
\end{align*}
\]

**Proof** (1) \( f \circ k = T[U[f] \circ U[k]] \) so that \( U[f \circ k] = U[T[U[f] \circ U[k]]] \). But \( U[f] \circ U[k] \) is an umbra and for sets which are umbra the umbra operation undoes the surface operation. Hence \( U[f \circ k] = U[T[U[f] \circ U[k]]] = U[f] \circ U[k] \).

(2) \( f \circ k = T[U[f] \circ U[k]] \) so that \( U[f \circ k] = U[T[U[f] \circ U[k]]] \). But \( U[f] \circ U[k] \) is an umbra and for sets which are umbra, the umbra operation undoes the surface operation. Hence,

\[
U[f \circ k] = U[T[U[f] \circ U[k]]] = U[f] \circ U[k]
\]

To illustrate how the umbra homomorphism property is used to prove relationships by first wrapping the relationship by re-expressing it in terms of umbra and surface operations and then transforming it through the umbra homomorphism property and finally by unwrapping it using the definitions of grayscale dilation and erosion, we state and prove the commutivity and associativity of grayscale dilation and the chain rule for grayscale erosion.

**Proposition** \( f \circ k = k \circ f \)

**Proof**

\[
\begin{align*}
f \circ k &= T[U[f] \circ U[k]] \\
&= T[U[k] \circ U[f]] \\
&= k \circ f
\end{align*}
\]

**Proposition** \( k_1 \circ (k_2 \circ k_3) = (k_1 \circ k_2) \circ k_3 \)

**Proof**

\[
\begin{align*}
k_1 \circ (k_2 \circ k_3) &= T[U[k_1] \circ U[k_2] \circ U[k_3]] \\
&= T[U[k_1] \circ (U[k_2] \circ U[k_3])] \\
&= T[(U[k_1] \circ U[k_2]) \circ U[k_3]] \\
&= T[U[k_1] \circ k_2 \circ U[k_3]] \\
&= (k_1 \circ k_2) \circ k_3
\end{align*}
\]

**Proposition** \( (f \circ k_1) \circ k_2 = f \circ (k_1 \circ k_2) \)

**Proof**

\[
\begin{align*}
(f \circ k_1) \circ k_2 &= T[U[f \circ k_1] \circ U[k_2]] \\
&= T[(U[f] \circ U[k_1]) \circ U[k_2]] \\
&= T[U[f] \circ (U[k_1] \circ U[k_2])] \\
&= T[U[f] \circ U[k_1 \circ k_2]] \\
&= f \circ (k_1 \circ k_2)
\end{align*}
\]

Relationships satisfied by gray dilation and erosion include

\[
\begin{align*}
f \circ g &= g \circ f \\
(f \circ g) \circ h &= f \circ (g \circ h) \\
(f \circ g) \circ h &= f \circ (g \circ h) \\
\max\{f, g\} \circ h &= \max\{f \circ h, g \circ h\} \\
\min\{f, g\} \circ h &= \min\{f \circ h, g \circ h\} \\
f < g &= f \circ h < g \circ h \\
f < g &= f \circ h < g \circ h \\
\max\{f, g\} \circ h &= \max\{f \circ h, g \circ h\} \\
-\circ g &= -\circ f \\
f \circ \max\{g, h\} &= \min\{f \circ g, f \circ h\}
\end{align*}
\]

The first three properties for grayscale dilation are algebraically similar to three properties of convolution.

\[
\begin{align*}
f \ast g &= g \ast f \\
(f \ast g) \ast h &= f \ast (g \ast h) \\
(f + g) \ast h &= (f \ast h) + (g \ast h)
\end{align*}
\]

This similarity strongly suggests the richness of the underlying algebraic structure for the grayscale morphological operations, despite the fact that they are highly non-linear.

Grayscale opening and closing are defined in an analogous way to opening and closing in the binary morphology and they have similar properties. The grayscale opening of \( f \) by structuring element \( k \) is denoted by \( f \circ k \) and is defined by \( f \circ k = (f \circ k) \circ k \). The grayscale closing of \( f \) by structuring element \( k \) is denoted by \( f \bullet k \) and is defined by \( f \bullet k = (f \bullet k) \circ k \).

There is a geometric interpretation to the grayscale opening and to the grayscale closing in the same manner that there is a geometric meaning to the binary morphological opening and closing. To obtain the opening of \( f \) by a paraboloid structuring element, for example, take the paraboloid, apex up, and slide it under all the surface of \( f \) pushing it hard up against the surface. The apex of the paraboloid may not be able to touch all points of \( f \). For example, if \( f \) has a spike narrower than the paraboloid, the top of the apex may only reach as far as the mouth of the spike. The opening is the surface of the highest points reached by some part of the paraboloid as it slides under all the surface of \( f \).

We have not mentioned the duality relationship between grayscale dilation and erosion. We need this in order to give the geometric interpretation to closing. The grayscale duality relationship is analogous to the binary duality relationship. Before stating it, we need the definition of grayscale reflection.
Definition Let \( f : F \to E \). The reflection of \( f \) is denoted by \( \bar{f} : E \to F \), and is defined by \( \bar{f}(x) = f(-x) \).

Grayscale Dilation Erosion Duality Theorem:

Let \( f : F \to E \) and \( k : K \to E \). Let \( x \in (F \ominus K) \cap (F \ominus \bar{K}) \) be given. Then \( -(f \ominus k)(x) = ((-f) \ominus \bar{k})(x) \).

It follows immediately from the gray scale dilation and erosion duality that there is a grayscale opening and closing duality.

Grayscale Opening and Closing Duality Theorem:

\[ -(f \circ k) = (-f) \circ \bar{k} \]

Having the grayscale opening and closing duality, we immediately have \( f \circ k = -((-f) \circ \bar{k}) \). In essence, this means that we can think of closing like opening. To close \( f \) with a paraboloid structuring element, we take the reflection of the paraboloid, turn it upside down (apex down), and slide it all over the top of the surface of \( f \). The closing is the surface of all the lowest points reached by the sliding paraboloid.

Relationships satisfied by gray scale opening and closing include the following:

\[
\begin{align*}
(f \circ g) \circ g &= f \circ g \\
(f \circ g) \circ g &= f \circ g \\
f \circ g &\leq f \\
f \leq f \circ g \\
f \leq g \Rightarrow f \circ h \leq g \circ h \\
f \leq g \Rightarrow f \circ h \leq g \circ h \\
-(-(f \circ g) = ((-f) \circ \bar{g}) \\
-(f \circ g) = ((-f) \circ \bar{g}) \\
f \circ g = (f \circ g) \circ g \\
(f \circ g) \circ g = (f \circ g) \circ g \\
(f \circ g) \circ g = (f \circ g) \circ g \\
\end{align*}
\]

5. Morphological Sampling Theorem

The preliminary part of this section sets the stage, discussing the appropriate morphological simplifying and filtering to be done before sampling. Certain relationships must be satisfied between the sampling set and the structuring element used for reconstruction. The main body of the section discusses two kinds of reconstructions of the sampled images: a maximal reconstruction accomplished by dilation and a minimal reconstruction accomplished by closing. Fundamental set bounding relationships are stated which indicate that the closing reconstruction of a set must be contained in the set itself which, in turn, must be contained in its dilation reconstruction. The closing reconstruction differs from the dilation reconstruction by just a dilation by the reconstruction structuring element, so the set bound relationships translate to geometric distance relationships. The section concludes by defining a suitable set distance function which measures the distance between the sampled set and the morphologically filtered set. The distance between the minimal reconstruction and the maximal reconstruction, and the distance between the morphologically filtered set and either of its reconstructions, are all less than the sampling distance.

The first conceptual issue which arises in developing a morphological sampling theorem is how to remove small objects, object protrusions, object intrusions and holes before sampling. It is exactly the presence of this kind of small detail before sampling which causes the sampled result to be unrepresentative of the original, just as in signal processing, the presence of frequencies higher than the Nyquist frequency causes the sampled signal to be unrepresentative of the original signal. This "aliasing" means that signals must be low pass filtered before sampling. Likewise in morphology, the sets must be morphologically filtered and simplified before sampling. Small objects and object protrusions can be eliminated by a suitable opening operation. Small object intrusions and holes can be eliminated by a suitable closing. Since opening and closing are duals, we develop our motivation by just considering the opening operation.

Opening a set \( F \) by a structuring element \( K \) in order to eliminate small details of \( F \) raises, in turn, the issue of how \( K \) should relate to the sampling set \( S \). If the sample points of \( S \) are too finely spaced, little will be accomplished by the reduction in resolution. On the other hand, if \( S \) is too coarse relative to \( K \), objects preserved in the opening may be missed by the sampling. \( S \) and \( K \) can be coordinated by demanding that there be a way to reconstruct the opened image from the sampled opened image. Of course, details smaller than \( K \), are removed by the opening and cannot be reconstructed.

One natural way to reconstruct a sampled opening is by dilation. If \( S \) and \( K \) were coordinated to make the reconstructed image (first opened, then sampled, and then dilated) the same as the opened image, we would have a morphological sampling theorem nearly identical to the standard sampling theorem of signal processing. However, morphology cannot provide a perfect reconstruction, as is illustrated by the following one-dimensional continuous domain example.

Let the image \( F \) be the union of three topologically open intervals

\[ F = (3.1, 7.4) \cup (11.5, 11.6) \cup (18.9, 19.8), \]

where \((x, y)\) denotes the topologically open interval between \( x \) and \( y \). We can remove all details of less than length 2 by opening with the structuring element \( K = (-1, 1) \).
consisting of the topologically open interval from -1 to 1. Then the opened image \( F \circ K = \{ 3.1, 7.4 \} \). What should the corresponding sample set be? Consider a sampling set \( S = \{ x \mid x \text{ an integer} \} \), with a sample spacing of unity; other spacings such as .2, .5 or .7 could illustrate the same sampling concept as well. The opened sample image \( (F \circ K) \cap S = \{ 4.5, 6, 7 \} \). Dilating by \( K \) to reconstruct the image produces \( \{(F \circ K) \cap S \} \circ K = \{ 3, 8 \} \), an interval which properly contains \( F \circ K \). The dilation fills in between the sample points, but cannot "know" to expand on the left end by a length of 9 and yet expand by 4 on the right end. However, the reconstruction is the largest one for which the sampled reconstruction \( \{(F \circ K) \cap S \} \circ K \cap S \) produces the opened opening \( (F \circ K) \cap S = \{ 4.5, 6, 7 \} \). This is easily seen in the example because substituting the closed interval \([3, 8]\) for the open interval \((3, 8)\) produces the sampled closed interval \([3, 8] \cap S = \{ 3, 4, 5, 6, 7, 8 \} \) which properly contains \( (F \circ K) \cap S = \{ 4.5, 6, 7 \} \).

The difficulty in reconstructing a sampled opened image morphologically can be understood in terms of the standard sampling theorem. Consider the case of a precise constant binary valued image. The required morphological simplification means that details smaller than \( K \) have been removed from all objects on the opened image, but this removal does not bandlimit the image. In fact the opened image belongs to a special class of infinite bandwidth signals, wherein reconstructing the sampled opened image as specified by the standard sampling theorem cannot produce the kind of aliasing found in Moiré patterns. The standard sampling theorem reconstruction produces a bandlimited signal which passes through the sample points. Thus, the step-like patterns, like the open intervals of \( F \), get reconstructed with ringing throughout and with overshoot and undershoot at step edges. By contrast, the morphological reconstruction cannot produce ringing, but the position of any step edge is uncertain within the sampling interval.

In the remainder of this section we give a complete derivation of the results illustrated in the example. First, note that to use a structuring element \( K \) as a "reconstruction kernel," \( K \) must be large enough to ensure that the dilation of the sampling set \( S \) by \( K \) covers the entire space \( E^N \). For technical reasons discussed in the derivations, we also require that \( K \) be symmetric, \( K = \tilde{K} \). In the standard sampling theorem, the period of the highest frequency present must be sampled at least twice in order to properly reconstruct the signal from its sampled form. In mathematical morphology, there is an analogous requirement. The sample spacing must be small enough that the diameter of \( K \) is just smaller than these two sample intervals. Hence, the diameter of \( K \) is large enough that it can contain two sample points but not three sample points. We express this relationship by requiring that

\[
x \in K_y \Rightarrow K_y \cap K_x \cap S \neq \emptyset \quad \text{and} \quad K \cap S = \{0\}.
\]

The first condition implies that the dilation of sample points fills the whole space; that is, \( S \circ K = E^N \) when \( K \) is not empty. If the points in the sampling set \( S \) are spaced no further than \( d \) apart, then the corresponding reconstructing kernel \( K \) could be the topologically open ball of radius \( d \) where the norm used to define distance is the \( L_{\infty} \) norm. In this case, \( x \in K_y \Rightarrow K_y \cap K_x \cap S \neq \emptyset \). Notice that two points which are \( d \) apart can lie on the diameter of \( K \). But since the ball is topologically open, the diameter cannot contain 3 points spaced \( d \) apart. Hence, the radius of \( K \) is just smaller than the sampling interval. Also notice that if a sample point falls in the center of \( K \), \( K \) will not contain another sample point.

We are now ready to state some propositions which lead to the binary morphological sampling theorem. In what follows, the set \( F \subseteq E^N \), the reconstruction structuring element will be denoted by \( K \subseteq E^N \), and the sampling set will be denoted by \( S \subseteq E^N \). Although not necessary for every relationship, we assume that \( S \) and \( K \) obey the following five conditions:

1. \( (1) S = S \circ S \)
2. \( (2) S = \tilde{S} \)
3. \( (3) K \cap S = \{0\} \)
4. \( (4) K = K \)
5. \( (5) a \in K_y \Rightarrow K_y \cap K_x \cap S \neq \emptyset \)

Figure 2 illustrates the \( S \) associated with a 3 to 1 downsampling. Figure 3 illustrates a structuring element \( K \) satisfying (3), (4) and (5). Since the dilation operation is commutative and associative, conditions (1) through (3) imply that the sampling set \( S \) with the dilation operation comprises an abelian group with the origin being its unit element. Thus, if \( x \in S \), then \( S_x = S \), and also since \( K \cap S = \{0\} \), \( x \in S \) implies \( K_x \cap S = \{x\} \). Both these facts are utilized in a number of the proofs to follow.

5.1 The Set Bounding Relationships

It is obvious that since \( 0 \in K \), the reconstruction of a sampled set \( F \cap S \) by dilation with \( K \) produces a superset of the sampled set \( F \cap S \). That is, \( F \cap S \subseteq (F \cap S) \circ K \). The reconstruction by dilation is open so that \( (F \cap S) \circ K = (F \cap S) \circ K \). Moreover, the erosion and dilation of the original image \( F \) by \( K \) bound the reconstructed sampled image in the sense of \( F \circ K \subseteq (F \cap S) \circ K \). Finally, the erosion of \( F \) by \( K \) and the dilation of \( F \) by \( K \) produce identical results: \( F \cap S = (F \cap S) \circ K \).

This first bounding relationship indicates that the reconstruction by dilation cannot be too far away from \( F \) since the reconstruction is constrained to lie between \( F \) and \( K \). Our next relationship strengthens the closeness between \( F \) and the dilation reconstruction \( (F \cap S) \circ K \). Sampling \( F \) and sampling the dilation reconstruction of \( F \) produce identical results: \( F \cap S = (F \cap S) \circ K \).
Considering sampling followed by the dilation reconstruction as an operation we discover that it is an increasing operation, distributes over union but not over intersection. That is:

1. \( F_1 \subseteq F_2 \) implies \( (F_1 \cap S) \circ K \subseteq (F_2 \cap S) \circ K \)
2. \( ((F_1 \cup F_2) \cap S) \circ K = [(F_1 \cap S) \circ K] \cup [(F_2 \cap S) \circ K] \)
3. \( ((F_1 \cap F_2) \cap S) \circ K \subseteq [(F_1 \cap S) \circ K] \cap [(F_2 \cap S) \circ K] \)

![Figure 2](image)

**Figure 2** illustrates sampling every third pixel by row and by column. The sampling set \( S \) is represented by all points which are shown as “*”.

![Figure 3](image)

**Figure 3** illustrates a symmetric structuring element \( K \) which is a digital disc of radius \( \sqrt{3} \). For the sampling set \( S \) of Figure 2, \( K \cap S = \emptyset \) and \( x \in K \), implies \( K \circ K \cap S \neq \emptyset \).

Our next relationship states that the dilation reconstruction of a sampled \( F \) is always a superset of \( F \) opened by the reconstruction structuring element \( K \). Hence, if \( F \) is open under \( K \), then \( F \) is contained in its dilation reconstruction: \( F \circ K \subseteq (F \cap S) \circ K \).

Thus the reconstruction of the opened sampled image \( F \circ K \) is bounded by \( F \circ K \) on the low side and \( F \circ K \) dilated by \( K \) on the high side.

\[ F \circ K \subseteq [(F \circ K) \cap S] \circ K \subseteq (F \circ K) \cap K \]

If \( F \) is morphologically simplified and filtered so that \( F = F \circ K \), then the previous bounds reduce to

\[ F \subseteq (F \cap S) \circ K \subseteq F \oplus K \]

By reconsidering our example \( F = (3.1, 7.4) \cup (11.5, 11.6) \cup (18.9, 19.8) \) which is not open under \( K = (-1, 1) \), we can see that such an \( F \) is not necessarily a lower bound for the reconstruction. In this case \( F \cap S = (4, 5, 6, 7, 19) \) and the reconstruction \( (F \cap S) \oplus K = (3, 8) \cup (18, 20) \), which does not contain \( F \). This suggests that the condition that \( F \) be open under \( K \) is essential in order to have \( F \subseteq (F \cap S) \circ K \).

We now state one last relation between the reconstruction \( (F \cap S) \circ K \) and \( F \). The reconstruction \( (F \cap S) \circ K \) is the largest open set which when sampled produces \( F \cap S \).

**Proposition**

Let \( A \subseteq E^2 \) satisfy \( A \cap S = F \cap S \) and \( A = A \circ K \). Then \( A \supseteq (F \circ K) \cap K \) implies \( A = (F \circ K) \cap K \).

**Proof**

Suppose \( A \supseteq (F \circ K) \cap K \) and \( A \cap S = F \cap S \) and \( A = A \circ K \). Since \( A \cap S = F \cap S, (A \cap S) \circ K = (F \circ K) \circ K \).

But \( A = A \circ K \) implies \( A \subseteq (A \cap S) \circ K = (F \circ K) \circ K \).

Now \( A \subseteq (F \circ K) \circ K \) together with the supposition \( A \supseteq (F \circ K) \circ K \) implies \( A = (F \circ K) \circ K \).

As the reconstruction \( (F \circ K) \circ K \) is maximal with respect to the two properties of being open and downsampling to \( F \cap S \), we are naturally led to ask about a minimal reconstruction. Certainly we would expect a minimal reconstruction to be contained in the maximal reconstruction and contain the sampled image. Since closing is extensive, we immediately have \( F \cap S \subseteq (F \circ K) \circ K \). Since \( 0 \in E \), erosion is an anti-extensive operation. Hence, \( (F \circ K) \circ K = [(F \circ K) \circ K] \circ K \subseteq (F \cap S) \circ K \) and these relations suggest the possibility of a reconstruction by closing. Indeed a closing reconstruction has set bounds similar to the dilation reconstruction: \( F \circ K \subseteq (F \cap S) \circ K \subseteq (F \cap S) \circ K \subseteq F \circ K \).

For true reconstruction, the sampled reconstruction should be identical to the sampled image. Indeed, this is the case \( [(F \cap S) \circ K] \cap S = F \cap S \).

Consider our example \( F = (3.1, 7.4) \cup (11.5, 11.6) \cup (18.9, 19.8) \), which is closed under \( K = (-1, 1) \). If the sampling set \( S \) is the integers then \( F \cap S = (4, 5, 6, 7, 19) \). Closing \( F \cap S \) with \( K \) can be visualized via the opening/closing duality \( (F \cap S) \circ K = ((F \cap S) \circ K) \circ K \). Opening the set \( (F \cap S) \circ K \) with \( K \) produces \( (F \cap S) \circ K = \{ x \neq 19 | x < 4 \text{ or } > 7 \} \). Hence \( (F \cap S) \circ K = ((F \cap S) \circ K) \circ K = \{ x | x \in 19 \text{ or } 4 \leq x \leq 7 \} \), and sampling produces \( [(F \cap S) \circ K] \cap S = (4, 5, 6, 7, 19) = F \cap S \).

From the previous relationship, it rapidly follows that sampling followed by a reconstruction by closing is an idempotent operation. That is, \( [(F \circ K) \circ K] \cap S = (F \cap S) \circ K \). A reconstruction by closing is obviously closed under \( K \). Moreover, it can be quickly determined that sampling followed by a closing reconstruction is increasing and does not necessarily distribute over union or intersection. That is,
\[ F_1 \subseteq F_2 \text{ implies } (F_1 \cap S) \bullet K \subseteq (F_2 \cap S) \bullet K \]
\[ ([F_1 \cup F_2] \cap S) \bullet K \supseteq ([F_1 \cap S] \bullet K) \cup ([F_2 \cap S] \bullet K) \]
\[ ([F_1 \cap F_2] \cap S) \bullet K \subseteq ([F_1 \cap S] \bullet K) \cap ([F_2 \cap S] \bullet K) \]

Furthermore, the closing reconstruction of a sampled \( F \) is always a subset of \( F \) closed by the reconstruction structuring element \( K \). That is, \( (F \cap S) \bullet K \subseteq F \bullet K \), so that \( (F \bullet K) \cap S \bullet K \subseteq F \bullet K \). Hence a closing reconstruction of an image which is closed before sampling will be a subset of the closed image.

By considering a simple example \( F = \{(0,1)\} \) which is not closed under \( K = (-1,1) \), we can see that \( F \) is not necessarily an upper bound for the reconstruction. In this case, \( F \cap S = \{0,1\} = F \) and the reconstruction \( (F \cap S) \bullet K = F \bullet K = [0,1] \) which properly contains \( F \). This suggests that the condition that \( F \) be closed under \( K \) is essential in order to have \( (F \cap S) \bullet K \subseteq F \).

Finally, we state one last relation between the reconstruction \( (F \cap S) \bullet K \) and \( F \). The reconstruction \( (F \cap S) \bullet K \) is the smallest closed set which when sampled produces \( F \cap S \).

5.2 Examples

To better illustrate the bounding relationships developed in the previous section between a set and its sample reconstructions, we show three simple examples. The domain of these examples is defined as \( E \times E \) where \( E \) is the set of integers. The sample set \( S \) is chosen as the set of even numbers in both row and column directions. Thus,

\[ S = \{(r,c) | r \in E \text{ and is even}; \ r \in E \text{ and is even}\}. \]

\( K \) is chosen as a box of size \( 3 \times 3 \) whose center is defined as the origin. The sets \( S, K \), and the three example sets \( F_1, F_2 \), and \( F_3 \) are shown in Figure 4. The sets \( F_1, F_2 \), and \( F_3 \) are \( 3 \times 3 \) boxes having different origins and the condition \( F = F \circ K \) holds for all these example sets.

The results of \( F \circ K, (F \cap S) \bullet K, (F \cap S) \circ K \), and \( F \circ K \) for sets \( F_1, F_2 \), and \( F_3 \) are shown in Figures 5, 6, and 7 respectively.

5.2.1 Example 1

All the pixels contained in the vertical boundaries of \( F_1 \) have even column coordinates and those in the horizontal boundaries of \( F_1 \) have even row coordinates. Since the sample set \( S \) consists of pairs of even numbers and \( F_1 \) is a \( 3 \times 3 \) box, the set \( F_1 \cap S \) consists of the four corner points of \( F_1 \) and is contained in the boundary set of \( F_1 \). Hence the closing reconstruction of \( F_1 \cap S \) recovers \( F_1 \) and the dilation reconstruction of \( F_1 \cap S \) is equivalent to \( F \circ K \). In fact, the following two equalities hold only when (1) the sampling is every other row and column, (2) a set's vertical boundaries have even column coordinates, and (3) its horizontal boundaries have even row coordinates

\[ (F \cap S) \bullet K = F \]
\[ (F \cap S) \circ K = F \circ K. \]

The bounding relationships for \( F_1 \), illustrated in Figure 5, are

\[ F_1 \circ K \subseteq (F_1 \cap S) \bullet K = F_1 \subseteq (F_1 \cap S) \circ K = F_1 \circ K. \]

5.2.2 Example 2

Since all pixels contained in the vertical boundaries of \( F_2 \) have odd column coordinates and those in the horizontal boundaries of \( F_2 \) have odd row coordinates and \( F_2 \) is a small \( 3 \times 3 \) box, the set \( F_2 \cap S \) does not contain any part of the boundary of \( F_2 \). Thus the closing reconstruction of \( F_2 \cap S \) equals \( F_2 \circ K \) and the dilation reconstruction of \( F_2 \cap S \) is equivalent to \( F_2 \). Similar to the example 1, the following equalities hold only when the sampling is every other row and column and has its odd column coordinates in its vertical boundaries and its odd row coordinates in its horizontal boundaries.
The bounding relationships for $F_2$, illustrated in Figure 6, are

$$F_2 \ominus K = (F_2 \cap S) \bullet K \subseteq F_2 = (F_2 \cap S) \oplus K \subseteq F_2 \oplus K.$$  

### 5.2.3 Example 3

The pixels contained in the vertical boundaries of $F_3$ have odd column coordinates and the pixels in the horizontal boundaries of $F_3$ have even row coordinates. Hence, no equalities should exist in the bounding relationship. This is illustrated in Figure 7. The bounding relationships for $F_3$ are

$$F_3 \ominus K \subseteq (F_3 \cap S) \bullet K \subseteq F_3 \subseteq (F_3 \cap S) \oplus K \subseteq F_3 \oplus K.$$  

Figure 6 shows a second example of how the erosion and dilation of $F_2$ bound the minimal reconstruction $(F_2 \cap S) \bullet K$ and the maximal reconstruction $(F_2 \cap S) \oplus K$, respectively, which in turn bound $F_2$.

To show why the opening condition $F = F \circ K$ is needed for the bounding relationships involving $F$, we show an example set $F_4$ which deviates from the set $F_3$ by adding six extra points to it (see Figure 8). The sample and reconstruction results of $F_4, F_4 \cap S, (F_4 \cap S) \bullet K$, and $(F_4 \cap S) \oplus K$ are exactly the same as the results for $F_3$. However, no bounding relationships between $F_4$ and its sample reconstructions are applicable. If we open $F_4$ by $K$, the bounding relationships exist because $F_4 \circ K = F_3$.

Figure 7 shows a third example of how erosion and dilation of $F_3$ bound (in this case properly) the minimal...
reconstruction \((F_3 \cap S) \circ K\) and the maximal reconstruction 
\((F_3 \cap S) \circ K\), respectively, which in turn bound (in this case properly) \(F_3\).

Figure 8 shows a set \(F_4\) which is not open under \(K\). Its sampling \(F_4 \cap S\) is identical to the sampling of \(F_3\) yet the maximal reconstruction \((F_4 \cap S) \circ K\) does not constitute an upper bound for \(F_4\) as in the previous examples.

5.3 The Distance Relationships

Having established the maximality of the reconstruction \((F \cap S) \circ K\) with respect to the property of being open and downsampling to \(F \cap S\), and the maximality of the reconstruction \((F \cap S) \circ K\) with respect to the property of being closed and downsampling to \(F \cap S\), we now give a more precise characterization of how far \(F \circ K\) is from \(F \circ K\), how far \(F \circ K\) is from \(F \circ K\), and how far \((F \cap S) \circ K\) is from \((F \cap S) \circ K\). This is important to know since \(F \circ K\) is \((F \cap S) \circ K\) when \(F = F \circ K\), and \((F \cap S) \circ K\) is \((F \cap S) \circ K\) when \(F = F \circ K\). Notice that in all three cases there is no difference between the lower and the upper set bound is just a dilation by \(K\). This motivates us to define a distance function to measure the distance between two sets and to work out the relation between the distance between a set and its dilation by \(K\) with the size of the set \(K\). In this section we show that with a suitable definition of distance, all these distances are less than the radius of \(K\). Since \(K\) is related to the sampling distance, all the above-mentioned distances are less than the sampling interval.

For the size of a set \(B\), denoted by \(r(B)\), we use the radius of its circumscribing disk. Thus, \(r(B) = \min_{x \in B} \max_{y \in B} \|x - y\|\). The more mathematically correct forms of inf for \min\ and sup for \max\ may be substituted when the space \(E\) is the real line. For a set \(A\) which contains a set \(B\), a natural pseudo-distance from \(A\) to \(B\) is defined by \(\rho(A, B) = \max_{a \in A} \min_{b \in B} \|a - b\|\). This pseudo distance satisfies (1) \(\rho(A, B) \geq 0\), (2) \(\rho(A, B) = 0\) implies \(A \subseteq B\), and (3) \(\rho(A, C) \leq \rho(A, B) + \rho(B, C) + r(B)\). The symmetric relation (2) is weaker than the corresponding metric requirement that \(\rho(A, B) = 0\) if and only if \(A = B\), and relation (3) is weaker than the metric triangle inequality.

The pseudo distance \(\rho\) has a very direct interpretation. \(\rho(A, B)\) is the radius of the smallest disk which when used as a structuring element to dilate \(B\) produces a result which contains \(A\).

**Proposition**
Let \(\text{disk}(r) = \{x : \|x\| \leq r\}\) and \(A, B \subseteq \mathbb{R}^N\). Then
\[
\max_{a \in A} \min_{b \in B} \|a - b\| = \inf_{A \subseteq B \subseteq \text{disk}(r)} |A|.
\]

**Proof**
Let \(\rho_0 = \max_{a \in A} \min_{b \in B} \|a - b\|\) and \(r_0 = \inf_{A \subseteq B \subseteq \text{disk}(r)} |A|\). Since \(A \subseteq B \subseteq \text{disk}(r_0)\), \(A\) and \(B\) are contained in the same disk. Suppose \(A, B \subseteq \text{disk}(r_0)\). Then \(\max_{a \in A} \min_{b \in B} \|a - b\| = 0\). Hence, \(\min_{a \in A} \max_{b \in B} \|a - b\| = 0\). Hence, \(\min_{a \in A} \max_{b \in B} \|a - b\| = 0\). Therefore,
\[
0 \geq \max_{a \in A} \min_{b \in B} \|a - b\| - \|b - y\| \geq \max_{a \in A} \min_{b \in B} \|a - b\| - \|b - y\|
\]
implies \(0 \geq \rho_0 - r_0\), so that \(\rho_0 \leq r_0\). Finally, \(r_0 \leq \rho_0\) and \(\rho_0 \geq r_0\).

The pseudo distance \(\rho\) can be used as the basis for a true set metric by making it symmetric. We define the set metric \(\rho_m(A, B) = \max(\rho(A, B), \rho(B, A))\), also called the Hausdorff metric. \(\rho_m\) satisfies \(\rho_m(A, B) = \inf_{r(A \subseteq B \subseteq \text{disk}(r))} |A|\). This happens since
\[
\rho_m(A, B) = \inf_{r(A \subseteq B \subseteq \text{disk}(r))} |A|\]
and \(B \subseteq A \subseteq \text{disk}(r)\).

A strong relationship between the set distance and the dilation of sets must be developed to translate set bounding relationships to distance bounding relationships. We show that \(\rho(A \subseteq B, C \subseteq D) \leq \rho_m(A, C) + \rho(B, D)\) and then quickly extend the result to \(\rho_m(A \subseteq B, C \subseteq D) \leq \rho_m(A, C) + \rho_m(B, D)\).
Proposition

(1) \( \rho(A \oplus B, C \oplus D) \leq \rho(A, C) + \rho(B, D) \)
(2) \( \rho_M(A \oplus B, C \oplus D) \leq \rho_M(A, C) + \rho_M(B, D) \)

Proof

(1)
\[ \rho(A \oplus B, C \oplus D) = \max_{x \in A \oplus B} \min_{y \in C \oplus D} \|x - y\| \]
\[ = \max_{x \in A} \max_{y \in B} \min_{z \in C} \min_{u \in D} \|x + z - y - u\| \]
\[ \leq \max_{x \in A} \max_{y \in B} \min_{z \in C} \|x + z - y - d\| \]
\[ \leq \max_{x \in A} \max_{y \in B} \min_{z \in C} \|x - c - y + d\| \]
\[ \leq \max_{x \in A} \max_{y \in B} \|x - c\| + \max_{z \in C} \min_{u \in D} \|u - d\| \]
\[ \leq \rho(A, C) + \rho(B, D) \]

(2)
\[ \rho_M(A \oplus B, C \oplus D) = \max(\rho(A \oplus B, C \oplus D), \rho(C \oplus D, A \oplus B)) \]
\[ \leq \max(\rho(A, C) + \rho(B, D), \rho(C, A) + \rho(D, B)) \]
\[ \leq \min(\rho(A, C), \rho(C, A)) + \max(\rho(B, D), \rho(D, B)) \]
\[ \leq \rho_M(A, C) + \rho_M(B, D) \]

From this last result, it is apparent that dilating two sets with the same structuring element cannot increase the distance between the sets. Dilation tends to suppress differences between sets, making them more similar. More precisely, if \( B = D = K \), then \( \rho_M(A \oplus K, C \oplus K) \leq \rho_M(A, C) \).

It is also apparent that \( \rho_M(A, A \oplus K) = \rho_M(A \oplus 0, A \oplus K) \leq \rho_M(A, A) + \rho_M(0, K) = \rho_M(0, K) \leq \max_{x \in K} \|x\| \).

Indeed, since the reconstruction structuring element \( K = \bar{K} \) and \( \bar{x} \in K \), the radius of the circumscribing disk is precisely \( \max_{x \in K} \|x\| \).

Hence, the distance between \( A \) and \( A \oplus K \) is no more than the radius of the circumscribing disk of \( K \).

Since \( \rho_M(A, A \oplus K) \leq \max_{x \in K} \|x\| \) and \( \max_{x \in K} \|x\| = r(K) \), we have \( \rho_M(A, A \oplus K) \leq r(K) \). Also, since \( A \ast K \supseteq A \), \( \rho_M(A \ast K, A) = \rho(A \ast K, A) \).

Hence, \( \rho_M(A \oplus K, A) = \rho(A \ast K, A) \leq \rho(A \ast K, A) = \rho(A, K, A) \leq r(K) \).

From this, it immediately follows that the distance between the minimal and maximal reconstructions, which differ only by a dilation by \( K \), is no greater than the size of the reconstruction structural element; that is, \( \rho_M((F \cap S) \ast K, (F \cap S) \ast K) \leq r(K) \).

When \( F = F \cup K = F \ast K \), then \( (F \cap S) \ast K \subseteq F \subseteq (F \cap S) \ast K \).

Since the distance between the minimal and maximal reconstruction is no greater than \( r(K) \) it is unsurprising that the distance between \( F \) and \( \ast \) or either of the reconstructions is no greater than \( r(K) \).

When the image \( F \) is open under \( K \), the distance between \( F \) and its sampling \( F \cap S \) can be no greater than \( r(K) \). Why? It is certainly the case that \( F \cap S \subseteq F \subseteq (F \cap S) \ast K \). Hence \( \rho_M(F, F \cap S) \leq \rho_M(F \cap S, (F \cap S) \ast K) \leq r(K) \).

If two sets are both open under the reconstruction structuring element \( K \), then the distance between the sets must be no greater than the distance between their samplings plus the size of \( K \).

From this last result it is easy to see that if \( F \) is closed under \( K \), then the distance between \( F \) and its minimal reconstruction \( (F \cap S) \ast K \) is no greater than \( r(K) \). Consider,

\[ \rho_M(F, (F \cap S) \ast K) \leq \rho_M(F \cap S, ((F \cap S) \ast K) \cap S) \]
\[ + r(K) = \rho_M(F \cap S, F \cap S) + r(K) = r(K) \]

These distance relationships mean that just as the standard sampling theorem cannot produce a reconstruction with frequencies higher than the Nyquist frequency, the morphological sampling theorem cannot produce a reconstruction whose positional accuracy is better than the radius of the circumscribing disk of the reconstruction structuring element \( K \). Since the diameter of this disk is just short of being large enough to contain two sample intervals, the morphological sampling theorem cannot produce a reconstruction whose positional accuracy is better than the sampling interval.

5.4 Examples

We use the example sets \( F_1, F_2, F_3 \), and \( F_4 \) in computing the distance between the original images and the sample reconstruction images. The value \( \max_{x \in K} \|x\| \) for each \( x \in K \) are shown in Figure 9. The minimum value among them, \( \sqrt{2} \), is the radius \( r(K) \) since \( r(K) = \min_{x \in K} \max_{x \in K} \|x - y\| \).

![Figure 9](image)

The \( \max_{x \in K} \|x - y\| \) values for all \( x \in K \), where \( K \) is the digital disc having radius \( \sqrt{2} \).

We now measure the distance between two sample reconstructions for all the example sets. To compute \( \rho_M((F_1 \cap S) \ast K, (F_1 \cap S) \ast K) \) we first compute \( \rho((F_1 \cap S) \cap K, (F_1 \cap S) \ast K) \) and \( \rho((F_1 \cap S) \ast K, (F_1 \cap S) \ast K) \). The values \( \min_{x \in (F_1 \cap S) \ast K} \|x - y\| \) for all \( x \in (F_1 \cap S) \ast K \) are shown in Figure 10. The maximum value among them, \( \sqrt{2} \), is the distance \( r(F_1, K) \) equals \( 0 \). Similarly, we can compute \( \rho((F_1 \cap S) \ast K, (F_1 \cap S) \ast K) \) which equals 0.

Thus, \( \rho_M((F_1 \cap S) \ast K, (F_1 \cap S) \ast K) \) equals \( \sqrt{2} \) which is exactly the radius \( r(K) \). Similarly, the distance between two reconstructions of sets \( F_2, F_3, F_4 \), can be measured and they are all equal to \( r(K) \).
Figure 10. The \( \min_{y \in (F \cap S) @ K} \|x - y\| \) for all \( x \in (F \cap S) @ K \).

What is the distance \( \rho_M((F \cap S) @ K) \) for the example sets? Since \( F = (F \cap S) @ K, \rho_M(F \cap S, (F \cap S) @ K) = \rho_M((F \cap S) @ K, (F \cap S) @ K) = r(K) \). It is easy to see \( \rho_M((F \cap S) @ K) = 0 \) because \( F = (F \cap S) @ K \). Figure 12 shows the values \( \min_{x \in (F \cap S) @ K} \|x - y\| \) for each \( x \in (F \cap S) @ K \), their maximum value being \( \rho(F \cap S) @ K, F = 1 \). Since \( F \subseteq (F \cap S) @ K, \rho(F \cap S) @ K) \) equals 0. Hence, \( \rho_M((F \cap S) @ K, F) = 1 \).

Note that since \( F \neq F @ K, \rho_M((F \cap S) @ K, F) \leq r(K) \).

Using the minimum reconstruction, the positional accuracy for the example sets are:

\[
\begin{align*}
\rho_M((F_1 \cap S) @ K) &= 0 < r(K) \\
\rho_M((F_2 \cap S) @ K, F_2 @ K) &= \sqrt{2} = r(K) \\
\rho_M((F_3 \cap S) @ K, F_3 @ K) &= 1 < r(K) \\
\rho_M((F_4 \cap S) @ K, F_4 @ K) &= 2 > r(K)
\end{align*}
\]

Also, since \( F \neq F @ K, \rho_M((F_1 \cap S) @ K, F_1 @ K) \leq r(K) \).

5.5 Binary Morphological Sampling Theorem

This section summarizes the results developed in the previous sections. These results constitute the binary digital morphological sampling theorem.

Let \( F, K, S \subseteq \mathbb{B}^N \). Suppose \( K \) and \( S \) satisfy the sampling conditions:

1. \( S @ S = S \)
2. \( S = \mathbb{B}^N \)
3. \( K \cap S = \{0\} \)
4. \( K = \mathbb{B}^N \)
5. \( x \in K \) implies \( F \cap S @ K \neq \emptyset \)

Then

1. \( F \cap S = [(F \cap S) @ K] @ S \)
2. \( F \cap S = [(F \cap S) @ K] @ S \)
3. \( F \cap S = K @ S \)
4. \( F @ K \subseteq (F \cap S) @ K \)
5. \( F = F @ K = F @ K, \text{ then } (F \cap S) @ K \subseteq F @ (F \cap S) @ K \)
6. \( F = F @ K \) and \( A @ S = F @ S, \text{ then } A @ S \subseteq (F \cap S) @ K \)
7. \( A = F \) and \( A @ S = F @ S, \text{ then } A @ S = (F \cap S) @ K \)
8. \( F = F @ K, \text{ then } \rho_M(F, (F \cap S) @ K) \leq r(K) \)
9. \( F = F @ K, \text{ then } \rho_M((F \cap S) @ K, F) \leq r(K) \)

6. Operating In the Sampled Domain

Section 5 established the relationship between the information contained in the sampled set and the information contained in the unsampled set. It shows that a minimal and maximal reconstruction can be computed from the sampled
set. When the set is smooth enough with respect to the sampling $S$ (that is, when the set is both open and closed under the reconstruction structuring element), then the minimal and maximal reconstructions bound the unsampled set, differing from it by no more than the sampling interval length.

Not addressed in Section 5 is the relationship between the computationally more efficient procedure of morphologically operating in the sampled domain versus the less computationally efficient procedure of morphologically operating in the unsampled domain. In this section we quantify just exactly how close a morphological operation in the sampled domain can come to the corresponding morphological operation in the original domain. Thus we answer the question of how to compute the largest length of sampling interval which can produce an answer close enough to the desired answer when morphologically operating in the sampled domain.

The first proposition states that a sampled dilation contains the dilation of the sampled sets and a sampled erosion is contained in the erosion of the sampled sets.

**Proposition**

Let $B \subseteq E^n$ be the structuring element employed in the dilation or erosion. Then

1. $(F \cap S) \oplus (B \cap S) \subseteq (F \oplus B) \cap S$
2. $(F \cap S) \oplus (B \cap S) \supseteq (F \oplus B) \cap S$

Unfortunately, the containment relations cannot, in general, be strengthened to equalities. But we can determine the conditions under which the equality occurs and we can determine the distance between sets such as $(F \cap S) \oplus (B \cap S)$, which is the dilation of the sampled sets, and $(F \oplus B) \cap S$, which is the sampling of the dilation. In the sampled domain, we can compare the scheme of sampling and then performing the dilation in the sampled domain to dilating first and then sampling. We also inquire about how different things could be in the unsampled domain by comparing performing the dilation in the sampled space and then reconstructing versus performing the dilation in the unsampled domain. The next proposition states that this difference in the sampled domain cannot be more than $2r(K)$.

**Proposition**

If $F = F \circ K$ and $B = B \circ K$, then $\rho_m((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)) \leq 2r(K)$

**Proof**

First consider $\rho((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)) \leq \rho(F \oplus B, (F \cap S) \oplus (B \cap S))$. Since $F = F \circ K$ and $B = B \circ K$, $F \subseteq (F \cap S) \circ K$ and $B \subseteq (B \cap S) \circ K$. Hence,

$$\rho((F \oplus B, (F \cap S) \circ K) \oplus (B \cap S)) \leq \rho((F \cap S) \circ K \circ (B \cap S)) \leq \tau(K \circ K) \leq 2r(K)$$

Next note that $\rho((F \cap S) \circ K \circ (B \cap S), (F \cap B) \cap S) = 0$. Since $(F \cap S) \circ (B \cap S) \subseteq (F \cap B) \cap S$. Now $\rho_m((F \cap S) \circ K \circ (B \cap S), (F \cap B) \cap S) = \max\{\rho((F \cap B) \cap S, (F \cap S) \circ (B \cap S)), \rho((F \cap S) \circ (B \cap S), (F \cap B) \cap S)\} \leq \max\{2r(K), r(0)\} = 2r(K)$

Whereas dilation tends to suppress differences, erosion tends to accentuate differences. Consider the following example. Let $F$ be a disk of radius 12 and $B$ be a disk of radius 10. Then $F \cap B$ is a disk of radius 2. Now define $F'$ to be a disk of radius 12 with its center point deleted. Notice that the pseudo set distance between $F$ and $F'$ is zero. But although $F'$ close to $F$, $F' \cap B = \emptyset$. The difference of one point makes all the difference.

More formally, consider the difference between the erosion of $F$ and the erosion of $F \circ K$.

$$\rho_m((F \circ K) \circ B, F \circ B) = \rho((F \circ B) \circ K, F \circ B) \geq \rho((F \circ B) \circ K, F \circ B)$$

since $(F \circ K) \circ B \subseteq (F \circ B) \circ K$ where $\rho((F \circ B) \circ K, F \circ B)$ is no greater than and could be as close to $\tau(K)$ as possible.

Thus we cannot expect that the difference between performing an erosion in the sampled domain versus performing a sampling of the erosion in the unsampled domain is no greater than the size of $K$. However, we do obtain set bounding relationships for dilation and erosion using the following relationships:

Dilating (eroding) a sampled set by a sampled structuring element is equivalent to sampling the dilation (erosion) of the unsampled set by the sampled structuring element.

1. $(F \cap S) \circ (B \cap S) = [F \circ (B \cap S)] \cap S$
2. $(F \cap S) \circ (B \cap S) = [F \circ (B \cap S)] \cap S$

Also, the dilation of the minimal reconstruction by a structuring element $B$ open under $K$ is contained in the dilation of the maximal reconstruction by the sampled structuring element $B \cap S$.

**Lemma**

Let $B = B \circ K$. Then $[(F \cap S) \circ K] \circ B \subseteq [(F \cap S) \circ K] \cap (B \cap S)$

**Proof**

Let $x \in [(F \cap S) \circ K] \circ B$. Then there exists an $f \in (F \cap S) \circ K$ and $b \in B$ such that $x = f + b$. Since $B = B \circ K$, $b \in B$ implies there exists a $y$ such that $b \in K_y \subseteq B$. But because of the sampling constraint between $K$ and $S$, $b \in K_y$ implies $K_y \cap S \neq \emptyset$. Therefore, $x \in [(F \cap S) \circ K] \circ B$. Hence, $[(F \cap S) \circ K] \circ B \subseteq [(F \cap S) \circ K] \cap (B \cap S)$. The proof is complete.
Therefore, there exists a \( z \in K_i \cap K_j \cap S \). Now \( z \in K_i \) implies that \( z = k + b \) for some \( k \in K \). Since it is also the case that \( z \in K_j \), it must be that \( z \in B \) because \( K_j \subseteq B \).

Recall that \( x = f + b = f + z - k = (f - k) + z \). Since \( f \in (F \cap S) \circ K = [(F \cap S) \circ K] \circ K \) and since \( -k \in K = K, f - k \in [(F \cap S) \circ K] \circ K = (F \cap S) \circ K \). Since \( z \in B \) and \( z \in S, z \in B \cap S \). Finally, \( -f - k \in (F \cap S) \circ K \) and \( z \in B \cap S \) imply \( x = (f - k) + z \in [(F \cap S) \circ K] \circ (B \cap S) \).

Now we see that dilation in the sampled domain and dilation in the unsampled domain are equivalent exactly when the structuring element \( B \) of the dilation is open under \( K \), and when the image \( F \) is its minimal reconstruction.

**Theorem**

Let \( B = B \cap K \). Then \( (F \cap S) \circ (B \cap S) = [(F \cap S) \circ K] \circ B \cap S \).

**Proof**

\[
(F \cap S) \circ (B \cap S) = [(F \cap S) \circ B] \cap S \text{ is always true.}
\]

Since \( F \cap S \subseteq (F \cap S) \circ K \),
\[
(F \cap S) \circ B \cap S \subseteq [(F \cap S) \circ K] \circ B \cap S \text{. But (}
[(F \cap S) \circ K] \circ B \subseteq [(F \cap S) \circ K] \circ (B \cap S) \text{ when } B = B \circ K.
\]

Hence, \( (F \cap S) \circ (B \cap S) \subseteq [(F \cap S) \circ K] \circ (B \cap S) \cap S \). Now \( [(F \cap S) \circ K] \circ (B \cap S) \cap S \subseteq [(F \cap S) \circ K] \circ (B \cap S) \cap S \). Hence \( (F \cap S) \circ (B \cap S) \subseteq (F \cap S) \circ K \cap (B \cap S) \cap S \).

Since \( (F \cap S) \circ K \cap (B \cap S) \cap S \subseteq (F \cap S) \circ K \cap (B \cap S) \cap S \) so that \( (F \cap S) \circ (B \cap S) \subseteq [(F \cap S) \circ K] \circ (B \cap S) \).

The equality relationship established in the theorem immediately leads to a set bounding relationship for dilation.

\[
[(F \circ K) \circ B] \cap S \subseteq [(F \cap S) \circ K] \circ B \cap S = (F \cap S) \circ (B \cap S) \subseteq (F \circ B) \cap S
\]

Also from the theorem, it becomes apparent that the difference between the maximally reconstructed dilation and the dilation of the minimal reconstruction can be no more than the size of \( K \) when \( B \) is open under \( K \). This happens because

\[
\rho_M([(F \cap S) \circ (B \cap S)] \circ K, [(F \cap S) \circ K] \circ B) \leq \rho_M([(F \cap S) \circ (B \cap S)] \circ K) \cap S, (F \cap S) \circ (B \cap S) \circ K) \cap S + r(K)
\]

Similarly, eroding a sampled by a sampled structuring element is equivalent to eroding the maximal reconstruction by the structuring element and then sampling when the structuring element is open under \( K \).

**Theorem**

Let \( B = B \circ K \). Then \( (F \cap S) \circ (B \cap S) = [(F \cap S) \circ K] \circ B \cap S \).

**Proof**

The sampling conditions imply \( [(F \cap S) \circ K] \cap B = F \cap S \).

Hence,

\[
(F \cap S) \circ (B \cap S) = [(F \cap S) \circ K] \circ (B \cap S) \subseteq [(F \cap S) \circ K] \circ (B \cap S) \cap S \subseteq [(F \cap S) \circ K] \circ (B \cap S)
\]

Under the sampling conditions, \( (F \cap S) \circ (B \cap S) \subseteq S \). So to complete the equality, we need to show that \( (F \cap S) \circ (B \cap S) \subseteq [[F \cap S] \circ K] \circ B \). Let \( x \in (F \cap S) \circ (B \cap S) \).

Then \( x \cap S \subseteq F \cap S \). Since \( B = B \circ K \), \( B \subseteq (B \cap S \circ K) \).

Hence \( B_x \subseteq (B \cap S \circ K) \subseteq (B \cap S) \). But \( (B \cap S) \subseteq (F \cap S) \circ K \). Now by definition of erosion, if \( B_x \subseteq (F \cap S) \circ K \), then \( x \subseteq [(F \cap S) \circ K] \circ B \).

This immediately leads to some set bounding relationships for erosion.

\[
(F \circ B) \cap S \subseteq [(F \cap S) \circ K] \circ B \cap S \subseteq [(F \circ K) \circ B] \cap S
\]

The previous theorem also makes it apparent that the difference between the maximally reconstructed erosion and the erosion of the maximal reconstruction can be no more than the size of \( K \) when both \( B \) and the erosion of the maximal reconstruction are open under \( K \). This happens because

\[
\rho_M([(F \cap S) \circ (B \cap S)] \circ K, [(F \cap S) \circ K] \circ B) \leq \rho_M([(F \cap S) \circ (B \cap S)] \circ K) \cap S, (F \cap S) \circ (B \cap S) \circ K) \cap S + r(K)
\]

Just as it was the case that dilating (eroding) a sampled set by a sampled structuring element is equivalent to sampling the dilation (erosion) of the unsampled set by the sampled structuring element, so it is also the case that opening (closing) a sampled set by a sampled structuring element is equivalent to sampling the opening (closing) of the unsampled set by the sampled structuring element. These relationships are useful in establishing when the opening and closing operations are equivalent in the sampled and unsampled domain.

(1) \( (F \circ (B \cap S)) \cap S = (F \cap S) \circ (B \cap S) \)

(2) \( (F \cap (B \cap S)) \cap S = (F \cap S) \circ (B \cap S) \)

The bounding relationships between the sampled and unsampled domains for the opening and closing operations now follow immediately.

**Theorem**

Suppose \( B = B \circ K \), then

(1) \( (F \circ (B \cap S)) \cap S \subseteq (F \cap S) \circ (B \cap S) \subseteq [[(F \cap S) \circ K] \circ B \cap S \)

(2) \( (F \cap (B \cap S)) \cap S \subseteq (F \cap S) \circ (B \cap S) \subseteq [[(F \cap S) \circ K] \circ B \cap S \)
(2) \( ([F \cap S] \ast K) \ast B) \cap S \subseteq (F \cap S) \ast ((B \cap S) \ast K) \cap S \)

Proof

(1) Notice that \( [(B \cap S) \ast K] \circ (B \cap S) = (B \cap S) \ast K \). Under this condition, \( (F \circ [(B \cap S) \ast K]) \cap S \subseteq [F \circ (B \cap S)] \cap S \). But by a previous proposition \( (F \circ (B \cap S)) \cap S = (F \cap S) \circ (B \cap S) \).

Now suppose \( x \in (F \ast B) \ast (B \cap S) \). Then there exists a \( y \) such that \( x \in (B \cap S) \ast y \subseteq F \cap S \). But \( (B \cap S) \ast y \subseteq (F \cap S) \ast K \) since dilation is an increasing operation. Hence, \( [(B \cap S) \ast K] \ast y \subseteq (F \cap S) \circ K \). Since \( B = B \circ K \), \( B \subseteq (B \cap S) \circ K \). Then, \( B \ast y \subseteq (F \cap S) \circ K \).

Also, \( x \in (B \cap S) \ast y \) implies \( x \in B \ast y \). Finally, \( x \in B \ast y \subseteq (F \cap S) \circ K \) implies \( x \in [(F \cap S) \circ K] \circ B \).

(2) By a previous proposition \( (F \cap S) \ast (B \cap S) = (F \ast (B \cap S)) \cap S \). Since \( [(B \cap S) \circ K] \circ (B \cap S) = (B \cap S) \ast K \), \( (F \ast (B \cap S)) \cap S \subseteq [(F \cap S) \circ K] \circ (B \cap S) \). Let \( R = (F \cap S) \circ K \). Since \( B = B \circ K \), \( R \ast B \) is open under \( K \). Hence \( R \ast B \subseteq [(R \ast B) \cap S] \circ K \). Now

\[
(R \ast B) \cap S = [(R \ast B) \cap B] \cap S \\
= ([((R \ast B) \cap S) \circ K] \circ B) \cap S
\]

But the sampled erosion of a maximal reconstruction is the erosion of the sampled set by the sampled structuring element. Hence,

\[
([(R \ast B) \cap S] \circ K) \circ B \cap S = [(R \ast B) \cap S] \circ (B \cap S)
\]

And the sampled dilation of a minimal reconstruction is the dilation of the sampled set by the sampled structuring element. Hence,

\[
[(R \ast B) \cap S] \circ (B \cap S) = [(R \ast B) \cap S] \circ (B \cap S)
\]

Finally, \( R \cap S = [(F \cap S) \ast K] \cap S = F \cap S \) so that \( [(F \cap S) \ast K] \ast B) \cap S \subseteq (F \cap S) \ast (B \cap S) \).

The bounding relationships immediately imply the following equivalence for the opening and closing operations between the sampled and unsampled domains.

**Theorem**

Suppose \( B = B \circ K \).

(1) If \( F = (F \cap S) \circ K \) and \( B = (B \cap S) \circ K \), then \( (F \cap S) \circ (B \cap S) = (F \circ B) \cap S \)

(2) If \( F = (F \cap S) \circ K \) and \( B = (B \cap S) \circ K \), then \( (F \cap S) \ast (B \cap S) = (F \circ B) \cap S \)

Proof

(1) If \( F = (F \cap S) \circ K \) and \( B = (B \cap S) \circ K \), the bounding relationship for opening becomes

\[
(F \circ B) \cap S \subseteq (F \cap S) \circ (B \cap S) \subseteq (F \circ B) \cap S
\]

from which we immediately obtain \( (F \circ B) \cap S = (F \cap S) \circ (B \cap S) \).

(2) If \( F = (F \cap S) \circ K \) and \( B = (B \cap S) \circ K \), the bounding relationship for closing becomes

\[
(F \circ B) \cap S \subseteq (F \cap S) \ast (B \cap S) \subseteq (F \circ B) \cap S
\]

from which we immediately obtain \( (F \circ B) \cap S = (F \cap S) \ast (B \cap S) \).

6.1 Examples

A simple example illustrates the bounding relationships of morphological operations operating in the pre- and post-sampled domain. The sample set \( S \) and the set \( K \) we used are those defined in the previous examples (see Figure 5). The sets \( F, B \), and \( K \) are defined in Figure 13. It is clear that \( B = B \circ K \). In Figure 14, we show the results of down-sampling every other row and every other column, \( F \cap S \), \( B \cap S \), and the sampled domain morphological operations, \( (F \circ S) \circ (B \cap S) \), \( (F \circ S) \circ (B \cap S) \). The results \( [(F \cap S) \circ K] \ast B \), \( [(F \cap S) \circ K] \ast B \), \( [(F \cap S) \circ K] \ast B \), and \( [(F \cap S) \circ K] \ast B \) are shown in Figure 15. Note that the following equalities hold:

\[
(F \cap S) \circ (B \cap S) = [(F \cap S) \circ K] \circ B \cap S
\]

and

\[
(F \cap S) \circ (B \cap S) = [(F \cap S) \circ K] \circ B \cap S.
\]

Figure 16 shows \( (F \circ B) \cap S \), \( (F \circ B) \cap S \), \( (F \circ (B \cap S)) \), and \( (F \circ (B \cap S)) \). Note that the following are true:

\[
(F \cap S) \circ (B \cap S) \subseteq (F \circ B) \cap S
\]

and

\[
(F \cap S) \circ (B \cap S) \subseteq (F \circ B) \cap S.
\]

It can be easily verified that

\[
(F \cap S) \circ (B \cap S) = [F \circ (B \cap S)] \cap S
\]

and

\[
(F \cap S) \circ (B \cap S) = [F \circ (B \cap S)] \cap S.
\]

In practical multiresolution image processing applications we would like to perform morphological operations in the sampled domain to reduce the computational expense. How well can a morphological operation be performed in the sampled domain rather than the original domain can be answered by the relationships and distances between \( (F \cap S) \circ (B \cap S) \) and \( (F \circ B) \cap S \) as well as \( (F \cap S) \circ (B \cap S) \) and \( (F \circ B) \cap S \). Unfortunately, the distance

\[
\rho_m((F \cap S) \circ (B \cap S), (F \circ B) \cap S) < 2 \rho(K)
\]
Figure 13 illustrates the sets $F$, $B$, and $K$. can be guaranteed only when $F = F \circ K$ and $B = B \circ K$. It can be very big when the set $F$ is not open. The set $F$ of Figure 13 is an example having a large difference between the pre- and post-sampled dilations because the conditions $F = F \circ K$ and $B = B \circ K$ are not satisfied.

We now show the distances between the pre- and post-sampled morphological operations. We first check the distance between $(F \cap S) \oplus (B \cap S)$ and $(F \oplus B) \cap S$. The \( \min_{y \in (F \oplus B) \cap S} \|x - y\| \) values for all $x \in (F \oplus B) \cap S$ are shown in Figure 17; their maximum value is \(\rho((F \oplus B) \cap S, (F \cap S) \oplus (B \cap S)) = 4\). Since $(F \cap S) \oplus (B \cap S) \subset (F \oplus B) \cap S$,
the distance \(\rho((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S) = 0\). Thus, \(\rho_M((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S) = 4\). Note that

\[ \rho_M((F \cap S) \oplus (B \cap S), (F \oplus B) \cap S) = 4 > 2\rho(K) \]

since $F \neq F \circ K$. Suppose $F' = F \circ K$ and $B' = B \circ K$. Figure 18 shows the results of $(F' \cap S) \oplus (B' \cap S)$ and $(F' \oplus B') \cap S$.

Since $(F' \cap S) \oplus (B' \cap S) = (F' \oplus B') \cap S$ in this example, we find that

\[ \rho_M((F' \cap S) \oplus (B' \cap S), (F' \oplus B') \cap S) = 0 < 2\rho(K). \]

Now we check the distances between the maximally reconstructed dilation (erosion) and the dilation (erosion) of the minimal (maximal) reconstruction, \(\rho_M([(F' \cap S) \oplus (B' \cap S)] \circ K, [(F' \cap S) \circ K] \oplus B)\) and \(\rho_M([(F \cap S) \circ K] \oplus B, [(F \cap S) \circ K] \oplus B)\). The values of \(\min_{x \in [(F \cap S) \oplus (B \cap S)] \circ K} \|x - y\| \) for all $x \in [(F \cap S) \oplus (B \cap S)] \circ K$ are shown in Figure 20; their maximum is \(\rho_M([(F \cap S) \circ K] \oplus B, [(F \cap S) \oplus K] \circ B) = 1\).

Since \([(F \cap S) \circ K] \oplus B \subset [(F \cap S) \ominus (B \cap S)] \circ K\), this implies \(\rho_M([(F \cap S) \circ K] \oplus B, [(F \cap S) \ominus (B \cap S)] \circ K) = 0\). Hence, \(\rho_M([(F \cap S) \circ K] \oplus B, [(F \cap S) \ominus (B \cap S)] \circ K) = 0\).

Figure 14. shows the results of sampling the $F$ and $B$ of Figure 13 and performing the dilation and erosion of $F \cap S$ by $B \cap S$ in the sampled domain.

Figure 15. shows the dilation and erosion of the minimal and maximal reconstruction of $F$ by the structuring element $B$ and also shows the sampling of this dilation and erosion.

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Figure 16 shows some morphological operations in the original domain followed by sampling.

1 \leq r(K). \quad \{ (F \cap S) \circ (B \cap S) \} \circ K \quad \text{and} \quad \{ (F \cap S) \circ K \} \circ B \quad \text{are shown in Figure 20. Note that} \quad \{ (F \cap S) \circ K \} \circ B \quad \text{is open under} \quad K. \quad \text{The values of} \quad \min_{y \in [F \cap S] \circ (B \cap S) \circ K} ||x - y|| \quad \text{for all} \quad x \in \{ (F \cap S) \circ (B \cap S) \} \circ K \quad \text{are shown in Figure 20; their maximum is} \quad \rho(\{ (F \cap S) \circ (B \cap S) \} \circ K). \quad \{ (F \cap S) \circ K \} \circ B \quad \text{is}] \quad \rho(\{ (F \cap S) \circ K \} \circ B), \quad \rho(\{ (F \cap S) \circ K \} \circ B), \quad \rho(\{ (F \cap S) \circ K \} \circ B) = 1. \quad \text{Since} \quad \rho(\{ (F \cap S) \circ K \} \circ B) \subseteq \rho(\{ (F \cap S) \circ (B \cap S) \} \circ K), \quad \text{this implies} \quad \rho(\{ (F \cap S) \circ K \} \circ B) = 0. \quad \text{Hence,} \quad \rho(\{ (F \cap S) \circ K \} \circ B), \quad \rho(\{ (F \cap S) \circ (B \cap S) \} \circ K), \quad \rho(\{ (F \cap S) \circ K \} \circ B), \quad \rho(\{ (F \cap S) \circ (B \cap S) \} \circ K) = 1 \leq r(K).

Figure 17 shows the values of \( \min_{x \in (F \circ B) \cap S} \) for all \( x \in (F \circ B) \cap S \).

Figure 18 shows the results of \( (F' \cap S) \circ (B' \circ S) \) and where \( F' = F \circ K \) and \( B' = B \circ K \). (See Figure 14 for the definition of \( F, B \), and \( K \)).

Next we show \( f_{il} \circ k \leq f \circ k \). By the umbra homomorphism theorem, \( U[f_{il} \circ b] = U[f] \circ (S \times E) \). Hence, \( U[f_{il} \circ b] = (U[f] \circ (S \times E) \circ b) \subseteq (U[f] \circ U[k]) \circ (S \times E) \circ U[k] \). But \( (S \times E) \circ U[k] = E^{n+1} \) if and only if \( S \circ K = E^{n+1} \). So \( U[f_{il} \circ b] \subseteq U[f] \circ U[k] = U[f \circ k] \) by the umbra homomorphism theorem. Hence, \( T[U[f_{il} \circ b]] \leq T[U[f \circ k]] \) which by definition of gray scale dilation implies \( f_{il} \circ k \leq f \circ k \). The analog of \( F \circ S = \{ [(F \cap S) \circ K] \cap S \} \) is \( f_{il} = (f_{il} \circ k)_{il} \). It holds under the condition that \( k(0) = 0 \).

In order to continue with the parallel development \( f \circ k \leq f_{il} \circ k \), we first need the stronger relation that for every \( x \in F \circ K \), there exists a \( s \in S \cap F \) such that \( x \in K \), and \( (f \circ k)(x) \leq f(s) + k(x - s) \). This result follows from the sampling condition \( u \in K \), implies \( S \cap K \cap K \neq \emptyset \) and a constraint on the structuring element \( k : k(a) \leq k(a + b) \) for every \( a, b \) satisfying \( a \in K, b \in K \) and \( a - b \in K \). This latter constraint is a new concept essential for the reconstruction structuring element in the gray scale morphology.

Before developing the proof for the inequality \( (f \circ k)(x) \leq f(s) + k(x - s) \) it will be useful to explore the meaning of the inequality \( k(a) \leq k(a - b) + k(b) \) since this is a constraint we have not had to deal with until now. The inequality \( k(a) \leq k(a - b) + k(b) \) together with \( k = k \) implies that \( k(y) \geq 0 \) for every \( y \in K \). This can easily be seen by letting \( a = x + y \) and \( b = x \). This leads to \( k(x + y) \leq k(x) + k(y) \). Then let \( a = x \) and \( b = x + y \). This leads to \( k(x) \leq k(x + y) + k(x + y) \). The two inequalities imply \( k(x) \leq k(y) + k(x + y) \leq (k(x) + 2k(y)) \) from which \( k(y) \geq 0 \) quickly follows.
The inequality \( k(a) \leq k(a - b) + k(b) \) also implies that for any integer \( n \geq 2 \), \( k(nz) \leq nk(x) \) for every \( z \in K \) satisfying \( mz \in K \) for every \( m, 2 \leq m \leq n \). The proof is by induction. Taking \( n = 2, a = 2z \) and \( b = z \) establishes the base case \( k(2x) \leq 2k(x) \). Suppose that for every \( m, 2 \leq m \leq n \), \( k(ma) \leq nk(a) \) and \( ma \in K \) and \( a \in K \). Taking \( e = (n+1)x \) and \( b = x \) produces \( k((n+1)x) \leq k(nx) + k(x) \). But \( k(nx) \leq nk(x) \). Hence, \( k((n+1)x) \leq nk(x) + k(x) = (n+1)k(x) \). Now by induction \( k(nx) \leq nk(x) \) for every integer \( n \geq 2 \) satisfying \( mz \in K \) for every \( m, 2 \leq m \leq n \).

\[
(F \cap S) \cup (B \cap S) \subseteq K
\]

\[
[(F \cap S) \cup K] \subseteq B
\]

Figure 19 illustrates how the distance between the result produced by reconstructing the morphological dilation done in the sampled domain and the dilation of the minimal reconstruction done in the original domain must be less than \( r(K) = \sqrt{2} \).

7. The Grayscale Sampling Theorem

In this section we present the extension of the morphological sampling theorem from the binary case to the grayscale case.

7.1 The Grayscale Bounding Relationships

The set bounding relationships for the binary morphology have a direct correspondence to function bounding relationships in gray scale morphology. In this section we develop the bounding relationships without spending much time or discussion even though the extensions are somewhat more involved. The gray scale analog to the relationship

\[
F \cap K \subseteq (F \cap S) \omega K \subseteq F \cap K \subseteq F \cap E
\]

where \( f_f : F \cap S \rightarrow E \) defined by \( f_f(x) = f(x) \) and it holds under very much the same conditions that the binary relationship holds. The only new requirement is for \( k \geq 0 \) which is stronger than the requirement that \( 0 \in U[K] \).

Finally, notice that \( k(a) \leq k(a - b) + k(b) \) for every \( a, b \in K \) satisfying \( a - b \in K \) and \( K = K \) imply, as well, \( k((b-a)+a) \leq k(b-a) + k(a) \) for every \( a, b \in K \) satisfying \( a - b \in K \).

Since \( k = k \), we obtain \( k(a-b) \geq \min(k(a)-k(b),k(b)-k(a)) \).

Constructing functions which satisfy the inequality is easy with the following procedure. Define \( k(0) = 0 \) and \( k(1) \) to be any positive number. Suppose \( k(m), m = 0, \ldots, n \) have been defined. Take \( k(n+1) \) to be any number satisfying

\[
\max\{k(u)-k(n+1-u)\} \leq k(n+1) \leq \min\{k(v)+k(n+1-u)\}
\]

After \( k \) is defined for all non-negative numbers in its domain define \( k(-n) = k(n) \) for \( n \geq 0 \).

The generating procedure works because \( k = k \) and the inequality implies \( k(x) - k(y-x) \leq k(x) + k(y-x) \). Hence, \( \max\{k(u)-k(y-u) \leq k(u) \} \leq \min\{k(v)+k(y-v)\} \).

Proposition

Let \( F, K, S \subseteq E^{n-1}, f : F \rightarrow E, k : K \rightarrow E \). Suppose \( u \subseteq K \) implies \( S \cap K, \cap K \neq \emptyset \) and \( k(a) \leq k(a-b) + k(b) \).

Then for every \( x \in F \cap K \), there exists an \( s \in S \cap F \) such that \( x \in K, a \) and \( (f \circ k)(x) \leq f(s) + k(x-a) \).

Proof

Let \( x \in F \cap K \). Then \( x \in F \cap K \) and \( (f \circ k)(x) \in U[f \circ k] \) such that \( (x, f \circ k(x)) \in U[f \circ k] \) and \( U[f \circ k] = U[f] \circ U[K] \). Hence, there exists \( u \subseteq E^{n-1} \times E \) such that \( (x, f \circ k(x)) \in U[f \circ k] \). Now \( (x, f \circ k(x)) \in U[f] \) implies \( (x, (f \circ k)(x)) \in U[K] \) which implies \( f(k(x)) - u \leq k(x-a) \). Thus \( u \leq f(k(x)) - k(x-a) \). But \( U[K], u \subseteq U[F] \) implies for every \( a \in K \), \( k(a) + u \subseteq U[F] \). Hence for every \( a \in K \), \( u \subseteq f(a) + u \subseteq f(s) + u \).

Now \( x \in K \) implies \( t \subseteq K \), \( s-t \subseteq K \) so that \( s-t \subseteq K \). And since \( s-t \subseteq K \), \( k(s-t) \leq f(s-t) = f(s) \). Now \( (f \circ k)(x) \leq f(s) - k(s-t) \). Thus \( f \circ k \leq f(s) - k(s-t) \).

Corollary

Let \( F, K, S \subseteq E^{n-1}, f : F \rightarrow E, k : K \rightarrow E \). Suppose \( u \subseteq K \) implies \( S \cap K, \cap K \neq \emptyset \) and \( k(a) \leq k(a-b) + k(b) \).

Then \( (f \circ k)(x) \leq f(s) + k(x-a) \).
Having determined that $f * k \leq f_i * k$, the maximality of $f_i * k$ comes easily.

**Proposition**

Let $G, F, K, S \subseteq E^{n-1}$, $g : G \to E$, $f : F \to E$. If $k(a) \leq k(a - b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$, then $g \circ k = g \circ k$ and $g \geq f_i * k$ implies $g = f_i * k$.

We continue our development with the bounding relations for the minimal gray-scale reconstruction.

**Proposition**

Let $F, K, S \subseteq E^{n-1}$, $f : F \to E$, and $k : K \to E$. If $k = k$, $k(a) \leq k(a - b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$, and $x \in K$, then $g \circ k = g \circ k$ implies $k \cap K \cap S = \emptyset$, then for every $u \in K$, $f(x + z) - k(x) \leq (f_i * k)(x) - k(x)$ for each $x \in K$.

**Proof**

Since $k(a) \leq k(a - b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$, then $f(x + z) - k(x) \leq f(x + z) - k(x)$ for every $u, z \in K$ satisfying $u - z \in K$.

Making a change of variables $t = u - z$, there results

$$f(x + z) - k(x) \leq [f(x + u - t) + k(t)] - k(u),$$

and

$$f(x + z) - k(x) \leq \max_{t \in \mathbb{K}} f_i(x + u - t) + k(t) - k(u) \leq (f_i * k)(x) - k(x)$$

for every $u \in K$.

**Proposition**

Let $F, K, S \subseteq E^{n-1}$, $f : F \to E$, and $k : K \to E$. Suppose $u \in K_i$, then $k \cap K \cap S = \emptyset$, $k = k$, and $k(a) \leq k(a - b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$.

Then for every $x \in F \cap K$, $(f \circ k)(x) \leq (f_i * k)(x)$.

**Proof**

Let $x \in F \cap K$. Then $(f \circ k)(x) = \min_{u \in K} (f(x + z) - k(x)) \leq (f_i \circ k)(x) - k(x)$ for every $u \in K$. Therefore,

$$(f \circ k)(x) \leq \min_{u \in K} ((f_i \circ k)(x) - k(x)) = (f_i * k)(x).$$

**Proposition**

Let $F, K, S \subseteq E^{n-1}$ and $f : F \to E$, $k : K \to E$. Then

(1) $f_i = (f_i * k)_i$,

(2) $f_i \leq f_i$.

**Proof**

Since closing is an increasing operation, $f_i \leq f_i * k$.

Since dilation is an increasing operation, $f_i * k \leq (f_i * k) \circ k = f_i \circ k$. Hence, $f_i \leq (f_i * k) \leq (f_i \circ k)$. But $(f_i \circ k) = f_i$, and this proves $f_i = (f_i * k)_i$.

(2) $U[f_i, k] = U[f_i] \cdot U[k]$ by the umbra homomorphism theorem. Also, $U[f_i] = U[f] \cap (S \times E)$ so that $U[f_i] \subseteq U[f]$. Hence, $U[f_i, k] \subseteq U[f] \cdot U[k] = U[f \circ k]$ by the umbra homomorphism theorem. This implies $f_i \leq f \circ k$.

As before, the maximality comes easy.

**Proposition**

Let $G, F, K, S \subseteq E^{n-1}$, $g : G \to E$, $f : F \to E$, and $k : K \to E$. Suppose $g_i = f_i$, and $g = g \circ k$. Then $g \leq f_i \leq k$ implies $g = f_i \leq k$.

7.2 The Grayscale Morphologic Sampling Theorem

This section summarizes the results developed in the previous sections. These results constitute the grayscale digital morphological sampling theorem.

Let $F, K, S \subseteq E^{n-1}$. Suppose $K$ and $S$ satisfy the sampling conditions

(1) $S \cap S = S$,

(2) $S = S$,

(3) $K \cap S = \emptyset$,

(4) $K = K$,

(5) $x \in K_i$ implies $K_i \cap K_i \cap S = \emptyset$. Let $f : F \to E$ and $k : K \to E$. Suppose further that $k$ satisfies

(6) $k = k$,

(7) $k(a) \leq k(a - b) + k(b)$ for every $a, b \in K$ satisfying $a - b \in K$.

(8) $k(0) = 0$.

Then

(1) $f_i = (f_i * k)_i$,

(2) $f_i = (f_i * k)_i$,

(3) $f_i \leq f_i \leq f_i$,

(4) $f_i \leq f \circ k$,

(5) $f = f \circ k$, then $f_i \leq f \leq f_i$,

(6) $g = g \circ k$, then $g \leq f_i \leq k$.

(7) $f \leq f_i$, then $g \leq f_i \leq k$.

7.3 Examples

The remaining examples illustrate how the morphological sampling theorem can lead to multiresolution processing techniques. The resolution hierarchy, called a pyramid, is produced typically by low pass filtering and then sampling to generate the next lower resolution level. Figure 21 shows a 5-level pyramid produced by pure sampling from a laser range map. The resolution level image size is $256 \times 256$. A 2-pixel wide line and a $4 \times 4$ box are placed intentionally at the upper right and upper left, respectively.

Figure 22 shows a 5-level morphological pyramid. At each level, the image is opened by a brick of size $3 \times 3$ and then sampled to generate the next lower resolution layer. Notice how the line in the upper right part of the image
has been eliminated. Figure 23 shows a similar 5-level morphological pyramid. In this pyramid the image at each level has been opened by a 3 x 3 brick, sampled, and then reconstructed, using the maximal reconstruction. The next lower resolution layer is generated by sampling as before.

**Figure 21** shows a 5-level pyramid of laser radar range image produced by pure sampling. The highest resolution image size is 256 by 256. A 2 pixel wide line segment and a 4 x 4 box are intentionally placed at the upper right and upper left of the image, respectively.

Figure 23 shows a 5-level pyramid of the same image as Figure 23. In each level, the image is opened by a brick of 3 x 3 and then is sampled and reconstructed before it is down sampled to generate the next level.

8. Research and Future Directions

The goal of our vision research group is to advance and develop the fundamental principles of computer vision by a systematic exploration of the required theoretical issues, the experimental issues, and the computer architecture issues. We do not approach theory for the sake of theory alone and we do not approach experiments for the sake of experiments and demonstrations alone. Rather, we employ statistically valid experiments using many images under a variety of conditions to explicitly or implicitly increase our understanding and insights about the constraints inherent in the reality with which computer vision deals. And we employ theory to build on the assumptions we currently hold about this validity. The technical material in the morphological sampling theorem which we described in this paper establishes a sound basis for doing multi-resolution morphology. Sound basis means that we now understand what the coarsest resolution is for which a shape can be looked for and found. Once we find the shape at the coarsest resolution, we understand that the corresponding shape at the next highest resolution is guaranteed to differ from the extrapolated shape at the lower resolution level by no more than the sampling interval, anywhere. These guarantees play a crucial role in the high-level knowledge base of the vision system we are building.

Our current research in mathematical morphology deals with understanding how to do adaptive morphology and to automatically construct optimal morphological processing sequences on the basis of a training set of examples. We intend to first solve the processing sequence problem in the restricted domain where we only have to work with 2D shape. Then we will extend it to 2D perspective projections of 3D objects.
An active area of research is concerned with making a variety of vision algorithms robust in a quantifiable way and characterizing the performance of the robust algorithm. We believe that with a good enough characterization of the performance of each subalgorithm in a composite vision algorithm, we can develop the error propagation methodology to characterize the performance of the composite vision algorithm on the basis of the performance of its participating algorithms.

Robust here means making the algorithm perform almost the same whether or not there is some fraction of blunders or gross errors in the data. Not only do robust algorithms behave well, but they naturally indicate which of their inputs they estimate to be reliable and which inputs they estimate to be errorful.

The robust algorithms with which we have begun are line, curve and surface fitting, as well as the pose estimation problem, in which it is required to determine a sensor or object rotation and translation when corresponding data point sets are given. Four pose estimation problems arise from the situations in which the corresponding data point sets are 2D to 2D, 3D to 3D, and 2D perspective projection to 3D, 2D perspective projection to 2D perspective projection. The first case arises when flat manufactured objects are being viewed. The second case arises with data from range finder sensors. The third arises from 3D vision from monocular views, and the fourth arises in two view motion and stereo.

Once the robust algorithms for pose estimation are established and we have a performance characterization of how many pairs of corresponding points in a data set can be outright blunders with only a slight degradation in performance, there will be enough information to integrate the pose estimation directly into the matching that establishes the corresponding point data sets. In this manner we expect to be able to develop an overall theory and methodology by which integrated information transfer between algorithms at different levels of a vision paradigm can take place.

In the high-level vision area we are continuing our earlier work on structural matching via the consistent labeling framework as well as investigating inference algorithms which are appropriate for high level control and which are computationally efficient. The inference algorithms we are developing are extensions of the approximate linear time propagate and divide inference algorithms we developed for propositional logic. In the structural matching area, we are exploring how known structure on the constraints in a matching problem can be utilized to speed up the matching algorithm for either exact or inexact matching. In this area we have already demonstrated a polynomial time matcher suitable for industrial vision.

In the manufacturing area we are working on developing a methodology to automatically produce a vision algorithm given an augmented CAD data base of the object to be recognized or inspected. Our current efforts here have been to establish a representation for 3D objects suitable for vision algorithms and get this representation implemented in a CAD software system. We also have done some preliminary work to analytically predict what features we expect to find in an image of the object under different lighting sources and different viewing angles.

Our group is quite interested in parallel architectures for vision. We are exploring, designing, and building pyramid and reconfigurable pipeline architectures for iconic pixel pushing algorithms, the intermediate iconic/symbolic algorithms, as well as higher level symbolic algorithms. Some of the most challenging architectural issues arise at the intermediate level where we are focusing on several representative problems:

1. determining where a given shaped object is;
2. processing with chain codes or run length encoded data;
3. Hough transforms with variable quantization and spatial clustering as a substitute for Hough transform;
4. generating and computing using topological image structures such as the winged data structure for arc segments, and their relationships.

For reconfigurable pipeline architectures we have already developed a new language called INSIGHT which lets the programs specify the desired parallelism. For the new architectural designs, we are working with other Electrical Engineering faculty whose main research areas are VLSI and parallel architectures.

References


Using Generic Geometric And Physical Models
For Representing Solids*

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Abstract

We present a solid modelling system based on powerful geometric and physical representations. The geometric representation combines constructive solid geometry (CSG) and boundary (Brep) representations. Solids are built as CSG trees by applying set operations to a very general set of primitives: generalized cylinders with either a straight or a curved axis. Using an efficient and robust boundary evaluator, we construct corresponding Breps in terms of trimmed surface patches and their boundaries. In addition to geometric models, our system describes the physical properties of material surfaces using very general physical models, unlike many existing systems, we accurately model the optical properties of both homogeneous and inhomogeneous materials. Our model for the scattering of light from the body of inhomogeneous materials based on modified Kubelka-Munk theory is significantly more general than previous models used in computer graphics and vision for this process. This is a bit surprising, since this process is largely responsible for the appearance of the vast majority of the objects we see in everyday life. We unify our physical and geometric representations using a method to map material surface properties onto our geometric models. Our representations allow for computationally efficient and physically accurate display of complex solids. We present several new rendering algorithms exploiting these advantages, including fast ray tracing for line drawing and shaded display. The system described has been implemented. Several examples demonstrate the effectiveness of our geometric and physical models.

Introduction

Being able to generate aesthetically pleasing images of a wide variety of objects requires expressive and robust geometric models and realistic yet possibly empirical models of the physical world. Conversely, being able to interpret complex images requires compact geometric models and a good understanding of the physics underlying the imaging process. In this work, we describe an implemented system which combines compact but expressive geometric models with accurate physical models of light sources and material surfaces.

The geometric modelling component of our system uses generalized cylinders as building blocks. We consider generalized cylinders with either a straight or a curved axis. They are well suited to visual tasks, as they combine both volume and surface information, and their shape is nicely "summarized" by their axis. These primitives are also very general, in particular, they include all the "natural" surfaces (cuboids, spheres, cylinders, cones, tori) used in most solid modelling systems. We have developed interactive tools for specifying the shape of generalized cylinders. Complex solids are built by combining these primitives through set operations. Our system allows this constructive solid geometry (CSG) representation to be built interactively, or from a simple modelling language. For many applications (e.g., prediction, display) however, it is necessary to have an explicit boundary representation (Brep) of solids. In the system presented here, it consists of a general description of objects' surfaces by graphs of trimmed surface patches separated by discontinuity curves. An efficient and robust boundary evaluator is used to transform the constructive solid geometry representation into the corresponding boundary representation. Powerful new rendering algorithms (including efficient ray tracing for line drawings and shaded display) have been developed which are capable of generating images of objects represented by our system. Figure 1 shows an example of such an object (a plastic valve).

We represent the physical properties of material surfaces using a generic physical modelling system. Our physical models are more general than any previously used models in several respects, yet these models are easy to use for image synthesis. Every parameter of our representation specifies an intrinsic physical property of a material surface. This is unlike many systems which model at the level of geometry dependent properties like reflectance. We represent a ma-