Mathematical Morphology and Computer Vision

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ABSTRACT
An algebraic system of operators, such as those of mathematical morphology, is useful for computer vision because compositions of its operators can be formed which, when acting on complex shapes, are able to decompose them into their meaningful parts and separate the meaningful parts from their extraneous parts. Such a system of operators and their compositions permit the underlying shapes to be identified and reconstructed as best possible from their distorted noisy forms. As well they permit each shape to be understood in terms of a decomposition, each entity of the decomposition being some suitably simple shape.

Since shape is a prime carrier of information in machine vision, there should be little surprise about the importance of mathematical morphology. Morphological operations can simplify image data preserving their essential shape characteristics and eliminate irrelevancies. As the identification and decomposition of objects, object features, object surface defects, and assembly defects correlate directly with shape, it is only natural that mathematical morphology has an essential structural role to play in machine vision.

1. Introduction
We begin our discussion of the algebra of morphology with a brief review of binary morphology. We will give examples illustrating how to use the algebra of binary morphology for shape primitive extraction. Then we will extend this with a discussion of grayscale morphology, generalized openings, morphological medians and serial derivatives bounding properties. Many of the omitted proofs can be found in Haralick et al. (1987).

1.1 Binary Morphology
The language of binary mathematical morphology is that of set theory. Those points in a set being morphologically transformed are considered as the selected set of points and those in the complement set are considered as not selected. In binary images, the selected set of points is the foreground and the set of points not selected is the background.

The primary morphological operations are dilation and erosion. From dilation and erosion the morphological operations of opening and closing can be composed. It is these latter two operations which have close connection to shape representation, decomposition, and primitive extraction.

1.2 Binary Dilation
The dilation of $A$ by $B$ is denoted by $A \oplus B$ and is defined by

$$A \oplus B = \{c \in E^N | c = a + b \text{ for some } a \in A \text{ and } b \in B\}$$

Because addition is commutative, dilation is commutative: $A \oplus B = B \oplus A$.

In practice, the sets $A$ and $B$ are not thought of symmetrically. The first set $A$ of the dilation $A \oplus B$ is associated with the image underlying morphologic processing and the second set $B$ is referred to as the structuring element, that shape which acts on $A$ through the dilation operation to produce the result $A \oplus B$. We will refer to $A$ as a set or as an image.

Dilation by disk structuring elements correspond to isotropic swelling or expansion algorithms common to binary image processing. Dilation by small squares ($3 \times 3$) is a neighborhood operation easily implemented by adjacency connected array architectures and is the one many image processing people know by the name “fill”, “expand”, or “grow”.

To characterize the dilation operation we need a notation for the translation of a set. Let $A$ be a subset of $E^N$ and $t \in E^N$. We denote the translation of $A$ by $t$ by $A_t$.

$$A_t = \{c \in E^N | c = a + t \text{ for some } a \in A\}$$

The dilation operation can be represented as a union of translates of the structuring element: $A \oplus B = \bigcup_{t \in A} B_t$.
This union of translates of the structuring element can be thought of like a neighborhood operator. The structuring element $B$ is swept over the image. Each time the origin of the structuring element $B$ touches a binary 1-pixel, the entire translated structuring element shape is ORed to the output image.

Because dilation is commutative, the dilation of $A$ with $B$ can also be represented as the union of the translates of the image $A$ taken over all the points of $B$: $A \oplus B = \bigcup_{t \in B} A_t$.
This representation is definitely not like a neighborhood operator. Rather, it corresponds to how pipeline architectures implement dilation in near real time. Neglecting, for the moment, boundary effects, translating the image corresponds to delaying the image by some amount in its raster scan order. The representation \( A \oplus B = \bigcup_{b \in B} A_b \) indicates that the dilation of the image \( A \) by \( B \) can be accomplished by delaying the raster scan of the image \( A \) by amounts corresponding to the points in structuring element \( B \) and then OR-ing the delayed raster scans to produce the dilation.

Because addition is associative, dilation is also associative: \((A \oplus B) \oplus C = A \oplus (B \oplus C)\). The associative property of dilation is called the chain rule or the iterative rule and has important practical significance since the dilation \((A \oplus B) \oplus C\) can be accomplished in a pipeline processor with a considerable saving over the equivalent dilation \(A \oplus (B \oplus C)\).

Other properties of dilation include that of dilating with a translated structuring element. This produces the translation of the dilation: \( A \oplus B = \{ a \oplus b \mid a \in A \} \). Because dilation is commutative, dilating a union of two sets is also the union of dilations: \((B \cup C) \oplus A = (B \oplus A) \cup (C \oplus A)\). The fact that dilation distributes over union is to mathematical morphology as the fact that convolution distributes over sums is to linear operator theory.

Dilating with a structuring element which is representable as a union of two sets is the union of the dilation, \( A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C) \). Dilating a set \( A \) with a structuring element containing the origin produces a result guaranteed to contain \( A \). Operators whose output contains their inputs are called extensive. Thus dilation is extensive when the structuring element contains the origin. However, when the origin is not in the structuring element, the resulting dilation may have nothing in common with the set being dilated. Finally, dilation preserves order. If \( A \subseteq B \), then \( A \oplus K \subseteq B \oplus K \). Operators having this property are called increasing.

### 1.3 Binary Erosion

The erosion of \( A \) by \( B \) is denoted by \( A \ominus B \) and is defined by

\[
A \ominus B = \{ x \in E^N \mid x + b \in A \text{ for every } b \in B \}
\]

The utility of the erosion transformation is better appreciated when the erosion is expressed in a different form. The erosion of an image \( A \) by a structuring element \( B \) is the set of all elements \( x \) of \( E^N \) for which \( B \) translated to \( x \) is contained in \( A \). In fact, this was the definition used for erosion by Matheron (1975). The proof is immediate from the definition of erosion and the definition of translation.

\[
A \ominus B = \{ x \in E^N \mid B_x \subseteq A \}
\]

Thus the structuring element \( B \) may be visualized as a probe which slides across the image \( A \), testing the spatial nature of \( A \) at every point. Where \( B \) translated to \( x \) can be contained in \( A \) (by placing the origin of \( B \) at \( x \)), then \( x \) belongs to the erosion \( A \ominus B \).

Erosion can be viewed as a morphological transformation which combines two sets using vector subtraction of set elements. Expressed as a difference of elements \( a \) and \( b \), erosion becomes \( A \ominus B = \{ x \in E^N \mid x = a - b \} \). This is the definition used for erosion by Haddow (1957).

Whereas dilation can be represented as a union of translates, erosion can be represented as an intersection of the negative translates:

\[
A \ominus B = \bigcap_{b \in B} A_{-b}
\]

This means that the same architecture which accomplishes dilation can accomplish erosion by changing the OR function to a AND function and using the image translated by the negated points of the structuring element instead of using the image translated by the points of the structuring element.

The erosion transformation is popularly conceived of as a shrinking of the original image. In set terms, the eroded set is often thought of as being contained in the original set. A transformation having this property is called anti-extensive. However, the erosion transformation is necessarily anti-extensive when the origin belongs to the structuring element. That is, if \( 0 \in B \), then \( A \ominus B \subseteq A \). To see this, let \( x \in A \ominus B \). Then, by definition of erosion \( x + b \in A \) for every \( b \in B \). Since \( 0 \in B \), \( x = x + 0 \in A \).

It is possible for an erosion of \( A \) by \( B \) to be a subset of \( A \) and for \( B \) also to not contain the origin. To see this, let \( A = \{ 1, 2, 3, 4 \} \) and \( B = \{-1, 1\} \). Then \( A \ominus B = \{ 2, 3 \} \subseteq A \) and yet \( 0 \notin B \).

Dilating a translated set results in a translated dilation. That is, \( A \ominus B = (A \ominus B) \). However, eroding with a translated structuring element results in a translated erosion where the translation is negated: \( A \ominus B = (A \ominus B) \). Like dilation, erosion is increasing: if \( A \subseteq B \), then \( A \ominus K \subseteq B \ominus K \). Further, eroding with a larger structuring element produces a smaller result: if \( K \subseteq L \), then \( A \ominus L \subseteq A \ominus K \).

The dilation and erosion transformations bear a marked similarity, in that what one does to the image foreground the other does to the image background. Indeed, their similarity can be formalized as a duality relationship. Two operators are dual when the negation of a formulation employing the first operator is equal to that formulation employing the second operator on the negated variables. An example is De Morgan's law, which states the duality between set union and intersection, \((A \cup B)' = A' \cap B'\). Here the negation of a set \( A \) is its complement, \( A' = \{ x \in E^N \mid x \notin A \} \). In morphology, negation of a set can occur in two possible ways: in the logical sense, in which case the negation is set complement, or in a geometrical sense, in which case it is a reversing of the orientation of the set with respect to its coordinate axes.
Such reversing is called reflection.

Let \( B \subseteq E^\times \). The reflection of \( B \) is denoted by \( \bar{B} \) and is defined by 
\[
\bar{B} = \{ x \mid \text{for some } b \in B, \ x = -b \}
\]
The reflection occurs about the origin. Matheron (1975) refers to \( \bar{B} \) as “the symmetrical set of \( B \) with respect to the origin”. Serra (1982) refers to \( \bar{B} \) as “\( B \) transpose.”

As given in the following theorem, the complement of an erosion is the dilation of the complement with the reflection: 
\[
(A \ominus B)^c = A^c \oplus \bar{B}.
\]
The erosion of a set with a structuring element which has a decomposition as a union of two sets is the intersection of the erosions: 
\[
A \ominus (K \cup L) = (A \ominus K) \cap (A \ominus L).
\]
However, the erosion of a set with a structuring element which has a decomposition as an intersection of two sets is only guaranteed to contain the union of the erosion of the set with each of the sets of the intersection decomposition: 
\[
A \ominus (B \cap C) \supseteq (A \ominus B) \cup (A \ominus C).
\]
Finally, with respect to structuring element decomposition, a chain rule for erosion holds when the structuring element is decomposable through dilation, 
\[
A \ominus (B \ominus C) = (A \ominus B) \ominus C.
\]

This relation is as important as the chain rule relation (associativity) for dilation because it permits a large erosion to be computed more efficiently by two or more successive smaller erosions.

Although dilation and erosion are dual, this does not imply that we can freely perform cancellation on morphological equalities. For example, if \( A = B \ominus C \), then dilating both sides of the expression by \( C \) results in \( A \ominus C = (B \ominus C) \ominus C \neq B \). However, a containment relationship does hold: 
\[
A \subseteq B \ominus C \implies A \subseteq B \ominus C
\]

The following list summarizes the algebraic relations between dilation, erosion, set union, and set intersection. The proofs for these relations are fairly direct.

\[
\begin{align*}
(A \oplus B) \ominus C &= A \ominus (B \ominus C) \\
(A \cup B) \ominus C &= (A \ominus C) \cup (B \ominus C) \\
A \ominus B &= \bigcup_{x \in B} A_x \\
A \subseteq B \Rightarrow A \subseteq C \subseteq B \subseteq C \\
A \ominus B \ominus C \subseteq (A \ominus C) \ominus (B \ominus C) \\
A \ominus (B \cup C) &= (A \ominus B) \cup (A \ominus C) \\
(A \oplus B)^c &= A^c \ominus B \\
A \ominus B = (A \ominus B)^c \\
A \ominus B \ominus C = A \ominus (B \ominus C) \\
(A \ominus B) \ominus C = (A \ominus C) \ominus (B \ominus C) \\
A \ominus B = \bigcap_{x \in B} A_x \\
\end{align*}
\]

1.4 Opening and Closing

Now we are ready to understand why dilation and erosion have an essential connection to shape. Dilations and erosions are usually employed in pairs, either dilation of an image followed by the erosion of the dilated result, or erosion of the image followed by the dilation of the eroded result. In either case, the result of successively applied dilations and erosions is an elimination of specific image detail smaller than the structuring element without the global geometric distortion of unsuppressed features. For example, opening an image with a disk structuring element smooths the contour, breaks narrow isthmuses, and eliminates small islands and sharp peaks or capes. Closing an image with a disk structuring element smooths the contours, fuses narrow breaks and long thin gulls, eliminates small holes, and fills gaps on the contours.

Of particular significance is the fact that image transformations employing successively applied dilations and erosions are idempotent, that is, their reaplication effects no further changes to the previously transformed result. The practical importance of idempotent transformations is that they comprise complete and closed stages of image analysis algorithms because shapes can be naturally described in terms of under what structuring elements they can be opened or can be closed and yet remain the same.

Opening and closing stand to morphology exactly as the orthogonal projection operator stands to linear algebra. An orthogonal projection operator is idempotent and selects that part of a vector that lies in a given subspace. Likewise, opening and closing provide the means by which given subs- shapes and supershapes of a complex shape can be selected.

The opening of image \( B \) by structuring element \( K \) is denoted by \( B \circ K \) and is defined by \( B \circ K = (B \ominus K) \ominus K \). The closing of image \( B \) by structuring element \( K \) is denoted by \( B \bullet K \) and is defined by \( B \bullet K = (B \ominus K) \ominus K \). If \( B \) is unchanged by opening it with \( K \), we say that \( B \) is open with respect to \( K \) or \( B \) is open under \( K \); while if \( B \) is unchanged by closing it with \( K \), then \( B \) is closed with respect to \( K \) or \( B \) is closed under \( K \).

The ability of an opening to select from a set that which matches the structuring element of the opening is immediate from the opening characterization theorem which states
\[
A \circ K = \{ x \in A \mid \exists t \in A \ominus K, x \in K, ~ \text{and} \ K \subseteq A \}
\]
The opening of \( A \) by \( K \) selects precisely those points of \( A \) which match \( K \) in the sense that the point can be covered by some translation of the structuring element \( K \) which itself is entirely contained in \( A \).

It is not difficult to understand how this characterization of opening arises. After all, by definition a point \( x \) is in the opening \( A \circ K \) if and only if \( x \in (A \oplus K) \preceq K \). And this happens if and only if for some \( t \in A \preceq K \), \( x \in (t \oplus K) \preceq K \). But \( t \in A \preceq K \) if and only if \( K \preceq A \).

It is also clear from the characterization theorem that unlike erosion and dilation, opening is invariant to the translation of the structuring element. That is, \( A \circ K = A \circ K_x \) for any \( x \). It is also easy to see from the opening characterization theorem that opening is an anti-extensive transformation. Like erosion and dilation, opening is also an increasing transformation.

Reorganizing the information in the opening characterization theorem, we can write

\[
(A \circ K)^c = [(A \oplus K)^c \ominus K]^c = (A \ominus K)^c \ominus K = (A \ominus K) \ominus K = A \ominus K
\]

The opening characterization theorem and the duality between opening and closing lead to a closing characterization which states

\[
A \bullet K = (x | x \in K) \cap A \neq \emptyset
\]

The closing of \( A \) includes all points satisfying the condition that anytime the point can be covered by a translation of the reflected structuring element, there is some point in common between the reflected translated structuring element and \( A \).

It is obvious from the closing characterization theorem that, like opening, closing is invariant to the translation of the structuring element. That is, \( A \circ K_x = A \circ K \) for any \( x \).

From the opening characterization theorem, it follows that the opening of \( A \) is contained in \( A \). Opening is anti-extensive. From the closing characterization theorem, it follows that \( A \) is contained in the closing of \( A \). Hence, closing is an extensive transformation. Since dilation and erosion are increasing operations, compositions of dilation and erosion will be increasing. In particular, then, closing as opening is an increasing operation.

Sets dilated by \( K \) remain invariant under an opening with \( K \). That is,

\[
A \preceq K = (A \oplus K) \circ K
\]

This comes about because \( A \preceq K \preceq A \) and since dilation is increasing, \((A \circ K) \preceq K \preceq A \preceq K\). But \((A \circ K) \preceq K = ((A \preceq K) \ominus K) \preceq K = (A \preceq K) \ominus K \preceq A \preceq K\).

Now from \((A \preceq K) \subseteq (A \circ K) \preceq K \subseteq (A \preceq K) \setminus A \preceq K\) we can infer that \( A \preceq K = (A \preceq K) \circ K \). By duality, eroded sets remain invariant under closing:

\[
A \preceq K = (A \ominus K) \bullet K
\]

The idempotency of the opening operation quickly follows from the opening representation theorem and the invariance of eroded sets to closing. Just note that

\[
(A \circ K) \circ K = \bigcup_{x \in A \circ K} K_x = \bigcup_{x \in A \circ K} K_x \preceq K
\]

By duality, closing is also idempotent.

The increasing and idempotency properties of opening and closing imply fairly directly that any set between the opening of \( A \) and \( A \) will have the same opening as \( A \) and any set between \( A \) and the closing of \( A \) will have the same closing as \( A \). That is, \( A \circ K \subseteq B \subseteq A \) implies \( B \circ K = A \circ K \) and \( A \subseteq B \subseteq A \bullet K \) implies \( B \bullet K = A \bullet K \).

Open sets are the smallest sets which have a given
erosion. To see this, suppose that the erosion of two sets \( A \) and \( B \) are identical; \( A \ominus K = B \ominus K \). Further, suppose that \( B \) purports to be a subset of the opening \( A \circ K \), \( B \subseteq A \circ K \). Then by dilating each side of \( A \ominus K = B \ominus K \) by \( K \) there results \( A \ominus K = B \ominus K \). Since \( B \circ K \subseteq B \) and \( A \circ K = B \circ K \), we have \( A \circ K \subseteq B \). But \( B \subseteq A \circ K \) and \( B \supseteq A \circ K \) implies \( B = A \circ K \). Hence \( B \) cannot be any smaller than \( A \circ K \). Similarly, closed sets are the largest sets which have a given dilation.

The fact that sets dilated by a structuring element \( K \) are open under \( K \) and sets eroded by a structuring element \( K \) are closed under \( K \) has some important consequences in terms of constructing other idempotent morphological operators. Consider, for example, the operator defined by first opening with one structuring element and then closing with another. Since opening and closing are increasing, this open-close composition is, of course, increasing. But it is also idempotent. Consider

\[
\left( (A \circ K) \circ L \right) \circ K = L
\]

\[
= \left( \left( (A \circ K) \ominus (K \circ L) \right) \oplus (K \circ L) \right) \circ K
\]

And since \( A \circ K \) must be open under \( K \circ L \),

\[
\left( (A \circ K) \circ L \right) \circ K = (A \circ K) \circ L
\]

In an exactly similar manner the close-open composition is idempotent:

\[
\left( (A \circ K) \circ L \right) \circ K = (A \circ K) \circ L
\]

It is straightforward to verify that \( A \) is open under \( K \) if and only if \( A \) can be represented as the dilation of some set by \( K \). Likewise, \( A \) is closed under \( K \) if and only if \( A \) can be represented as the erosion of some set by \( K \).

Unions of sets morphologically open with respect to \( K \) are morphologically open with respect to \( K \) and intersections of sets morphologically closed with respect to \( K \) are morphologically closed with respect to \( K \). Open sets which are dilated by any structuring element remain open. Closed sets which are eroded by any structuring element remain closed.

Opening with a structuring element \( K \) which can be expressed as a dilation decomposition \( K = K_1 + K_2 \) does produce the relationship \( A \circ (K_1 + K_2) \subseteq A \circ K_1 \). This is easily derived using the fact that opening is anti-extensive.

\[
A \circ (K_1 + K_2) = (A \circ K_1) \ominus K_2
\]

\[
= \left( ((A \circ K_1) \ominus K_2) \ominus K_1 \right) \ominus K_2
\]

\[
= \left( ((A \circ K_1) \ominus K_2) \ominus K_1 \right) \ominus K_2
\]

\[
= (A \circ K_1) \ominus K_2 = A \circ K_1
\]

However, if \( L \subseteq K \), it does not necessarily follow that \( A \circ L \supseteq A \circ K \). Similarly, closing with a structuring element \( K \subseteq K_1 \) must produce a superset of closing with \( K_1 \).

There is a theorem which establishes that if \( K \circ L = K \), then opening with the larger structuring element \( K \) does indeed result in a smaller opening, \( A \circ K \subseteq A \circ L \) and as well \( K \subseteq A \circ L \). There is also a sieve theorem which under the condition \( K \circ L = K \) provides that \( (A \circ K) \circ L = A \circ K \).

Here opening can be thought of as sieving. Opening with the larger structuring element \( K \) corresponds to sieving with the smaller sieve. Opening with the smaller structuring element \( L \) corresponds to sieving with the larger sieve. The operation \( (A \circ K) \circ L \) then corresponds to taking the material which has successfully passed through the holes of the smaller sieve and then passing it through the sieve with the larger holes. Obviously, everything which passed through the smaller holes will also pass through the large holes. Hence, the second sieving removes nothing so that \( (A \circ K) \circ L = A \circ K \).

It is similar with closing. \( K \circ L = K \) implies \( (A \circ K) \circ L = A \circ K \).

2. Generalized Openings and Closings

Many of the properties of morphological openings and closings arise from the fact that morphological openings are increasing, anti-extensive, and idempotent, and morphological closings are increasing, extensive and idempotent. In this section we introduce the generalized opening which is any operation having the increasing, anti-extensive, and idempotent properties. The generalized closing is any operation having increasing, extensive, and idempotent properties. It is fairly direct to see that unions of generalized openings are generalized openings. Let \( \gamma_1 \) and \( \gamma_2 \) be generalized openings. For any set \( A \), define \( \gamma \) by \( \gamma(A) = \gamma_1(A) \cup \gamma_2(A) \).

Let \( A \subseteq B \). Then

\[
\gamma(A) = \gamma_1(A) \cup \gamma_2(A)
\]

Since \( \gamma_1 \) and \( \gamma_2 \) are increasing, \( \gamma_1(A) \subseteq \gamma_1(B) \) and \( \gamma_2(A) \subseteq \gamma_2(B) \). Thus,

\[
\gamma(A) \subseteq \gamma_1(B) \cup \gamma_2(B) = \gamma(B)
\]

so that \( \gamma \) is increasing. \( \gamma \) is also anti-extensive. This follows from the anti-extensivity of \( \gamma_1 \) and \( \gamma_2 \).

\[
\gamma(A) = \gamma_1(A) \cup \gamma_2(A) \subseteq A \cup A = A
\]

Finally \( \gamma \) is idempotent. To see this, notice that

\[
\gamma(\gamma(A)) = \gamma(\gamma_1(A) \cup \gamma_2(A))
\]

\[
= \gamma_1(\gamma_1(A) \cup \gamma_2(A)) \cup \gamma_2(\gamma_1(A) \cup \gamma_2(A))
\]

\[
= \gamma_1(\gamma_1(A)) \cup \gamma_2(\gamma_2(A))
\]

\[
= \gamma(\gamma_1(A)) \cup \gamma(\gamma_2(A)) = \gamma(A)
\]

Also, since \( \gamma \) is anti-extensive, \( \gamma(\gamma(A)) \subseteq \gamma(A) \); \( \gamma(\gamma(A)) \supseteq \gamma(A) \) imply \( \gamma(\gamma(A)) = \gamma(A) \).
There are examples of generalized openings and closings that are not constructed from morphological openings and closings. The operator \( \gamma(A) = (A \ominus K) \cap A \) for any symmetric \( K \) is a generalized opening. Since \( A \ominus K = \{ x \in A \mid x \not\in K \} \), and \( K = \bar{K} \) the operator \( \gamma(A) \) selects all those points of \( A \) such that \( K \) hits \( A \). That \( \gamma \) is increasing and anti-extensive is direct. To prove that \( \gamma \) is idempotent, consider

\[
\gamma(\gamma(A)) = \left\{ \left( \left( A \ominus K \right) \cap A \right) \cap A \right\} = \left\{ \left( \left( \left( (A \ominus K) \cap A \right) \cap A \right) \right\} = \left\{ \left( \left( \gamma(A) \right) \right) \right\}
\]

As unions of generalized openings are generalized openings, intersections of generalized closings are generalized closings.

3. Gray Scale Morphology

The binary morphological operations of dilation, erosion, opening and closing are all naturally extended to gray scale imagery by the use of a min or max operation.

3.1 Grayscale Dilation and Erosion

We begin with the concepts of top surface of a set and the umbra of a surface. Suppose a set \( A \) in Euclidean \( N \)-space is given. We adopt the convention that the first \((N-1)\) coordinates of the \( N \)-tuples of \( A \) constitute the spatial domain of \( A \) and the \( N^\text{th} \) coordinate is for the surface. For grayscale imagery, \( N = 3 \). The top or top surface of \( A \) is a function defined on the projection of \( A \) onto its first \((N-1)\) coordinates. For each \((N-1)\)-tuple \( x \) the top surface of \( A \) at \( x \) is the highest value \( y \) such that the \( N \)-tuple \((x, y) \in A \).

Let \( A \subseteq E^N \) and \( F = \{ (x \in E^{N-1}) \mid \text{for some } y \in E, (x, y) \in A \} \). The top or top surface of \( A \), denoted by \( T[A] : F \rightarrow E \), is defined by

\[
T[A](x) = \max \{ y \mid (x, y) \in A \}
\]

A set \( A \subseteq E^{N-1} \times E \) is an umbra if and only if \((x, y) \in A \) implies that \([x, z] \in A \) for every \( z \leq y \). For any function \( f \) defined on some subset \( F \) of Euclidean \((N-1)\)-space the umbra of \( f \) is a set consisting of the surface \( f \) and everything below the surface.

Let \( F \subseteq E^{N-1} \) and \( f : F \rightarrow E \). The umbra of \( f \), denoted by \( U[f] \), \( U[f] \subseteq F \times E \), is defined by

\[
U[f] = \{ (x, y) \in F \times E \mid y \leq f(x) \}
\]

Obviously, the umbra of \( f \) is an umbra.

Having defined the operations of taking a top surface of a set and the umbra of a surface, we can define grayscale dilation. The gray scale dilation of two functions is defined as the surface of the dilation of their umbra.

Let \( F, K \subseteq E^{N-1} \) and \( f : F \rightarrow E \) and \( k : K \rightarrow E \). The dilation of \( f \) by \( k \) is denoted by \( f \circ k \) and is defined by

\[
f \circ k = T[U[f] \circ U[k]]
\]

The definition of grayscale dilation tells us conceptually how to compute the gray scale dilation, but this conceptual
way is not a reasonable way to compute it in hardware. The following theorem establishes that grayscale dilation can be accomplished by taking the maximum of a set of sums. Hence, grayscale dilation has the same complexity as convolution. However, of doing the summation of products as in convolution, a maximum of sums is performed.

Let \( f : F \rightarrow E \) and \( k : K \rightarrow E \). Then \( f \circ k : F \circ K \rightarrow E \) can be computed by

\[
(f \circ k)(x) = \max_{z \in \mathbb{R}} \{ f(x - z) + k(z) \}
\]

Figure 2 illustrates the calculation for the grayscale dilation. The definition for grayscale erosion proceeds in a similar way to the definition of grayscale dilation. The grayscale erosion of one function by another is the surface of the binary erosions of the umbra of one with the umbra of the other.

Let \( F \subseteq E^{\mathbb{R}} \) and \( K \subseteq E^{\mathbb{R}} \). Let \( f : F \rightarrow E \) and \( k : K \rightarrow E \). The erosion of \( f \) by \( k \) is denoted by \( f \circ k \). \( f \circ k : F \circ K \rightarrow E \), and is defined by

\[
f \circ k = T[U[f] \circ U[k]]
\]

Computing a grayscale erosion is accomplished not by performing the operation indicated in its definition; rather by taking the minimum of a set of differences. Hence its complexity is the same as dilation. Its form is like correlation with the summation of correlation replaced by the minimum operation and the product of correlation replaced by a subtraction operation.

Let \( f : F \rightarrow E \) and \( k : K \rightarrow E \). Then \( f \circ k : F \circ K \rightarrow E \) can be computed by

\[
(f \circ k)(x) = \min_{z \in K} \{ f(x + z) - k(z) \}
\]

The basic relationship between the surface and umbra operations is that they are, in a certain sense, inverses of each other. More precisely, the surface operation will always undo the umbra operation. That is, the surface operation is a left inverse to the umbra operation as given: \( T[U[f]] = f \).

However, the umbra operation is not an inverse to the surface operation. Without any constraints on the set \( A \), the strongest statement which can be made is that the umbra of the surface of \( A \) contains \( A \). When the set \( A \) is an umbra, then the umbra of the surface of \( A \) is itself \( A \). In this case the umbra operation is an inverse to the surface operation.

Having established that the surface operation is always an inverse to the umbra operation and that the umbra operation is the inverse to the surface operation when the set being operated on itself is an umbra, we next need to notice that the dilation of one umbra by another is an umbra and that the erosion of one umbra by another is also an umbra.

Now, we are ready for the umbra homomorphism theorem which states that the operation of taking an umbra is a homomorphism from the gray scale morphology to the binary morphology.

**Umbra Homomorphism Theorem:** Let \( F, K \subseteq E^{\mathbb{R}} \) and \( f : F \rightarrow E \) and \( k : K \rightarrow E \). Then

\[
\begin{align*}
(1) & \quad U[f \circ k] = U[f] \circ U[k] \\
(2) & \quad U[f \circ k] = U[f] \circ U[k]
\end{align*}
\]

**Proof:**

\[
\begin{align*}
(1) & \quad f \circ k = T[U[f] \circ U[k]] \\
& \quad U[f \circ k] = U[T[U[f] \circ U[k]]] \\
& \quad U[f] \circ U[k] = U[U[f] \circ U[k]] \\
(2) & \quad f \circ k = T[U[f] \circ U[k]] \\
& \quad U[f \circ k] = U[T[U[f] \circ U[k]]] \\
& \quad U[f] \circ U[k] = U[U[f] \circ U[k]]
\end{align*}
\]

To illustrate how the umbra homomorphism property is used to prove relationships by first wrapping the relationship by re-expressing it in terms of umbra and surface operations and then transforming it through the umbra homomorphism property and finally unwrapping it using the definitions of grayscale dilation and erosion, we state and prove the chain rule for grayscale erosion.

**Proposition:** \( f \circ k = k_2 \circ f \equiv (k_1 \circ k_2) \)

**Proof:**

\[
(f \circ k_1) \circ k_2 = T[U[f \circ k_1] \circ U[k_2]] = T[U[f] \circ U[k_1] \circ U[k_2]] = T[U[f] \circ U[k_1] \circ U[k_2]] = (f \circ k) \circ (k_1 \circ k_2)
\]

**Grayscale opening and closing are defined in an analogous way to opening and closing in the binary morphology and they have similar properties.**
Let \( f : F \rightarrow E \) and \( k : K \rightarrow E \). The grayscale opening of \( f \) by structuring element \( k \) is denoted by \( f \circ k \) and is defined by \( f \circ k = (f \triangleright k) \setminus k \).

Let \( f : F \rightarrow E \) and \( k : K \rightarrow E \). The grayscale closing of \( f \) by structuring element \( k \) is denoted by \( f \bullet k \) and is defined by \( f \bullet k = (f \triangleleft k) \circ k \).

The duality between gray scale dilation and gray scale erosion leads to a duality between gray scale opening and gray scale closing.

\[-(f \circ k)(x) = ((-f) \bullet k)(x).\]

It follows from the umbrum homomorphism theorem that

\[U[f \circ k] = U[f] \circ U[k] \quad \text{and} \quad U[f \bullet k] = U[f] \bullet U[k].\]

\[f \circ k = f \text{ if and only if } U[f \circ k] = U[f] \circ U[k] \quad \text{and} \quad f \bullet k = k \text{ if and only if } U[f \bullet k] = U[f] \bullet U[k].\]

A gray scale geometric representation for gray scale opening can be quickly determined using the above relation. Certainly it is the case that \((f \circ k)(x) = T[U[f \circ k]](x)\). But \(U[f \circ k] = U[f] \circ U[k]\) and by the opening representation theorem

\[U[f] \circ U[k] = \bigcup_{x \in U[f] \circ U[k]} U[k].\]

Hence,

\[(f \circ k)(x) = T \left[ \bigcup_{x \in U[f] \circ U[k]} U[k] \right](x).\]

The gray scale opening of \( f \) by \( k \) can be visualized by sliding \( k \) under \( f \). The locus of all the highest points reached by some part of \( k \) during the slide then constitutes the opening.

A geometric representation for closing can also be obtained. To set this up, notice that \((f \bullet k)(x) = T[U[f \bullet k]](x) = T[U[f] \bullet U[k]](x)\) by the umbrum homomorphism theorem. Now by the duality between opening and closing, \((f \bullet k)(x) = T[U[f] \circ U[k]](x)\). Using the convention that the top of a set \( A \) is a function whose domain consists of only those \( x \) for which \( \max(y(x, y) \in A) < \infty \), a concrete interpretation can be given to the to the complement of a set. It is just the bottom of the set. Denoting the bottom or bottom surface of a set \( A \) by \( B[A], B[A](x) = \min(y(x, y) \in A) \) where we use the convention that the domain for the function \( B[A] \) is precisely that set of all \( x \) such that \( \min(y(x, y) \in A) > -\infty \). Then we can represent the gray closing by

\[(f \bullet k)(x) = B[U[f] \circ U[k]](x) = B[\bigcup_{x \in U[f] \circ U[k]} U[k]](x)\]

The interpretation of the gray scale closing of \( f \) with \( k \) is to take the structuring element \( k \), left-right reflect it and turn it upside-down and sweep the result above the top of \( f \). The locus of all the lowest points reached by some part of the reflected upside-down structuring element during its sweep is the closing of \( f \) by \( k \).

From the above representation theorems for gray scale opening and closing it is apparent that \( f \circ k < f \) and \( f < f \bullet k \).

Also, since gray scale dilation and erosion are increasing, gray scale opening and closing are increasing. That is, \( f < g \) implies \( f \circ k < g \circ k \) and \( f \bullet k < g \bullet k \).

The important idempotency property of the opening and closing operation in binary morphology extends to gray scale morphology. The proof proceeds by expressing the opening or closing as the top of its umbrum. That is, it unwraps the umbra of the opening or closing, using the umbrum homomorphism theorem. The idempotency of the opening or closing in binary morphology reduces the inside expression of umbras. Finally, the umbrum homomorphism theorem rewraps the resulting expression as the umbra of a gray scale dilation or erosion. Therefore, \((f \circ k) \circ k = f \circ k\) and \((f \bullet k) \bullet k = f \bullet k\).

There are some important equalities and inequalities which relate umbras, tops, and mins and maxes. They can all be verified directly. The umbra of the minimum of two functions is the intersection of their umbras. The top of a union of two sets is the max of their tops. The top of an intersection of two sets is not greater than the minimum of the tops. To see why the last relation must be an inequality, consider the example sets \( A = \{(1, 4), (1, 5), (2, 3), (2, 4)\} \) and \( B = \{(1, 4), (1, 6), (2, 3), (2, 5)\}\). Then \( A \cup B = \{(1, 4), (2, 3)\}\) so that \( T[A \cap B](1) = 4 \) and \( T[A \cap B](2) = 3 \). But \( T[A](1) = 5 \), \( T[A](2) = 4 \), \( T[B](1) = 6 \), and \( T[B](2) = 5 \). Hence \( \min(T[A], T[B])(1) = 5 \) and \( \min(T[A], T[B])(2) = 5 \), thereby illustrating a case where \( T[A \cap B] < \min(T[A], T[B]) \).

Using the relations

\[U[f \circ g] = U[f] \cap U[g],\]

\[U[f \triangleleft g] = U[f] \cup U[g],\]

\[T[A \cup B] = \max(T[A], T[B]),\]

\[T[A \cap B] \leq \min(T[A], T[B]),\]

and the umbrum homomorphism theorems and the duality relations there follows

\[\max(f \circ g) \circ k = \max(f \circ k, g \circ k),\]

\[\min(f \circ g) \circ k \leq \min(f \circ k, g \circ k),\]

\[\max(f \bullet g) \circ k = \max(f \circ k, g \circ k),\]

\[\min(f \bullet g) \circ k \leq \min(f \circ k, g \circ k),\]

\[\max(f \circ g) \circ k \geq \max(f \bullet k, g \bullet k),\]

\[\min(f \circ g) \circ k \geq \min(f \bullet k, g \bullet k).\]

If \( f \circ k = f \) and \( g \circ k = g \), then \( \max(f \circ g) \circ k = \max(f, g) \).

If \( f \bullet k = f \) and \( g \bullet k = g \), then \( \min(f \circ g) \circ k = \min(f, g) \).
4. Openings, Closings and Medians

One of the most common nonlinear noise smoothing filters in image processing is the median filter. Its advantages include its being a high efficiency estimator of local mean when the noise is fat-tailed, its robustness to outlier pixel values, and its leaving edges sharp. Images which remain unchanged after being median filtered are special. They are called median root images. To obtain the "closest" median root image of a given input image just requires repeatedly median filtering the given image until there is no change. There is a relationship between openings, closings and median roots: an image which is both opened and closed with respect to a constant valued structuring element is a median root image.

To understand how this comes about, first notice that if $K$ is any subset of $E^{n-1}$ and $k$ is a constant valued structuring element defined on $K$, then the opening and closing of an image $f$ have a simple form:

$$(f * k)(x) = \min_{y \in K} \max_{z \in K} f(x - z + y)$$

$$(f \circ k)(x) = \max_{y \in K} \min_{z \in K} f(x - z + y).$$

And because of this the actual value of a constant valued structuring element does not influence the results of an opening or closing taken with respect to it.

From the expression of an opening or closing in terms of min and max it is apparent that the effective neighborhood used in an opening or closing with a structuring element whose domain is $K$ is $K \cap \hat{K}$. As we will be comparing an opening and closing with a median filtering, we will use $K \supseteq \hat{K}$ as the neighborhood for the median filtering. We denote the median value around the point $x$ of all points in the neighborhood $K \supseteq \hat{K}$ by $\text{med}_{x \in K} f(x + w)$. Since the opening at a point $x$ of $f$ with respect to a constant valued structuring element $k$ is given by $\text{max}_{y \in K} \text{min}_{z \in K} f(x - z + y)$, we focus our attention on the highest possible value of $\text{min}_{y \in K} \text{max}_{z \in K} f(x - z + y)$ as $z$ is allowed to vary over $k$. Suppose $|K| = M$. Now regardless of the value of $z$ the highest possible value will occur when the list of $M$ values $< f(x - z + y) : y \in K >$ contains the $M$ highest values of the list $< f(x + w) : w \in K \supseteq \hat{K} >$, which is guaranteed to contain every value of the list $< f(x - z + y) : y \in \hat{K} >$ when $0 \in K$. If the number of points in $K \supseteq \hat{K}$ is less than or equal to $2M - 1$ then the smallest among these $M$ highest values must be less than or equal to $\text{med}_{x \in K \supseteq \hat{K}} f(x + w)$ since the list $< f(x + w) : w \in K \supseteq \hat{K} >$ has no more than $2M - 1$ values. Then $(f * k)(x) \leq \text{med}_{x \in K \supseteq \hat{K}} f(x + w)$. A parallel argument leads to the fact that $\text{med}_{x \in K \supseteq \hat{K}} f(x + w) \leq (f \circ k)(x)$.

These two inequalities it immediately follows that if $f$ is both opened and closed with respect to $k$, then

$$(f \circ k)(x) = f(x) = (f \circ k)(x).$$

Hence functions which are both opened and closed with respect to a constant valued structuring element defined on a domain $K$ containing the origin and satisfying $|K \supseteq \hat{K}| \leq 2|K| - 1$ must be their own median roots.

There are some properties of opening and closing that suggest a morphological way of approximating a median filtering. We provide this motivation by considering functions defined on a one-dimensional domain. We fix a point $x$ and consider under what conditions is it the case for a particular $x$ that $f(x)$ might equal $(f * k)(x)$ or $(f \circ k)(x)$ where $k$ is a constant-valued structuring element.

From the geometric meaning of opening and closing, it is obvious that if $f(x)$ is monotonically increasing or monotonically decreasing over a large enough interval around $x$, then $f(x) = (f * k)(x) = (f \circ k)(x)$. How large is large enough? If the constant-valued structuring element has domain $K$, then large enough is $K \supseteq \hat{K}$. To see this, suppose the domain $K$ is some small interval. In this case, $f$ is monotonically increasing the left end point of the constant-valued structuring element will be able to touch $f(x)$ with the entire structuring element able to stay at or below $f$ for its entire length. If $f$ is monotonically decreasing, the right end point of the constant-valued structuring element will be able to touch $f(x)$ with the entire structuring element able to stay at or below $f$. Now it is clear from the geometric meaning of opening that $f(x) = (f * k)(x)$ when $f$ is monotone at $x$. A similar argument establishes that $f(x) = (f \circ k)(x)$ when $f$ is monotone at $x$.

Monotonicity is not the only sufficient cause of the value of $f$ at a given point being the value of its closing or opening at that given point. If $f(x)$ is the global minimum in a neighborhood of sufficient size around $x$, then $f(x) = (f * k)(x)$. Here sufficient size again means $(K \supseteq \hat{K})$. Note that the condition is global minimum and not relative or local minimum. As illustrated in Figure 3, if a relative minimum occurs between two other relative minima whose distance apart is less than the size of the structuring element domain, then the opened value at the middle relative minimum will be the lower of the surrounding and lower relative minimum. Similarly, if $f(x)$ is the global maximum in a neighborhood of sufficient size around $x$, then $f(x) = (f \circ k)(x)$.

Finally, if $f$ is monotone in a neighborhood of $K \supseteq \hat{K}$ around $x$ note that $f(x) = \text{med}_{x \in K \supseteq \hat{K}} f(x + w)$.

These properties about openings, closings and medians suggest the following morphological approximation $g$ to a median filtering: Define $g$ by $g(x) = (f * k)(x)$ if $(f * k)(x) = (f \circ k)(x) \geq \| f \circ k(x) - f(x) \|$ and $(f \circ k)(x)$ otherwise.

Consider the relationship of $g$ to the median filtered value. If $f$ is monotone at $x$, $\text{med}_{x \in K \supseteq \hat{K}} f(x + w) = (f \circ k)(x) = (f \circ k)(x)$ so that $g(x) = \text{med}_{x \in K \supseteq \hat{K}} f(x + w)$.
monotone at $x$ and $f$ has a global neighborhood minimum at $x$, then $(f \circ k)(x) = f(x) \leq \operatorname{med}_{w \in B} f(x+w)$ which implies that $g(x) = (f \circ k)(x) \geq \operatorname{med}_{w \in B} f(x+w)$. Hence the value of such minimum points are changed to values which are higher than the neighborhood median. However, the new value is not the neighborhood maximum since $f(x)$ cannot simultaneously be a neighborhood minimum, a neighborhood maximum, and not monotone at $x$. Thus a change has been made for the extreme value to something which is not an extreme value.

Likewise, if $f$ is not monotone at $x$ and if $f$ has a global neighborhood maximum at $x$, then $(f \circ k)(x) = f(x) \geq \operatorname{med}_{w \in B} f(x+w)$ which implies that $g(x) = (f \circ k)(x) \leq \operatorname{med}_{w \in B} f(x+w)$. Hence the value of such maximum points are changed to values which are lower than the neighborhood median. However, the new value is not the neighborhood minimum. Thus again a change has been made from an extreme value to something which is not an extreme value.

5. Bounding Second Derivatives

Opening or closing a gray scale image simplifies the image complexity. When the structuring element of the opening or closing is parabolic, this simplification can be understood in terms of the way the Laplacian of the opened or closed images are bounded. In this manner opening and closing can be viewed as morphologic filters which eliminate all points on the image having too high values of second partial derivatives. Actually, this elimination only occurs for pixels at which the first partial derivatives are suitably small. Opening bounds the Laplacian of the image from below and closing bounds it from above.

A precise statement of the bounding effect can be given as follows: Let the neighborhood support of the structuring element $k$ be a square $K \times K$ neighborhood and let the parabolic structuring element $k$ be defined by $k(x,y) = \frac{1}{2}(K^2 - x^2 - y^2)$. Let the image be denoted by $f$. Suppose the bound $B$ satisfies

$$
\max_{-K \leq x \leq K} \max_{-K \leq y \leq K} \max_{0 \leq t \leq 1} \left| R(x,y,s,t) \right| = B
$$

where

$$
f(x+s,y+t) = f(x,y) + \frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)
+ \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial x\partial y}(x,y) + R(x,y,s,t).
$$

If the first partial derivatives of $f$ are bounded as

$$
-KC \leq \frac{\partial f}{\partial x}(x,y) \leq KC
$$

and

$$
-KC \leq \frac{\partial f}{\partial y}(x,y) \leq KC
$$

and $f = (f \circ k) \circ k$ then the Laplacian is bounded by

$$
-2C - 2B \leq \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y).
$$

To understand how this arises, we first need to demonstrate the validity of:

If $-KC \leq W \leq KC$, then for every $z_1$, $-KC \leq z_1 \leq K$, there exists a $y_0$. If $y_0 - z_1 = 1$ and $-KC \leq y_0 \leq K$, such that $(z_1 - y_0)(W + z_0) \geq 0$.

This relation is proved by a straightforward case analysis.

Case 1: $z_1 = K$. In this case take $y_0 = K - 1$. Then $z_1 - y_0 = (W + z_0)C = -(K - (K - 1))(W + KC)$. Since $-KC \leq W$, then $W + KC \geq 0$.

Case 2: $z_1 = -K$. In this case take $y_0 = -K + 1$. Then $(z_1 - y_0)(W + z_0)C = -(K - (K - 1))(W - KC)$. Since $W \leq KC$, then $-(W - KC) \geq 0$.

Case 3: $K < z_1 < K$. In this case examine $W + z_0C$. Either $W + z_0C \geq 0$ or $W + z_0C < 0$. If $W + z_0C \geq 0$, take $y_0 = z_0 - 1$. If $W + z_0C < 0$, take $y_0 = z_0 + 1$.

With this preliminary relation proved, we can proceed with the derivation of the main bounding relation. Since $f = (f \circ k) \circ k$,

$$
f(x,y) = \max_{-K \leq x \leq K} \max_{-K \leq y \leq K} \left( f(x + v_1 - u_1, y + v_2 - u_2) + k(v_1, v_2) + k(u_1, u_2) \right).
$$

But

$$
f(x + v_1 - u_1, y + v_2 - u_2) = f(x,y)
+ \frac{(v_1 - u_1)^2}{2} \frac{\partial f}{\partial x^2}(x,y)
+ \frac{(v_2 - u_2)^2}{2} \frac{\partial f}{\partial y^2}(x,y)
+ (v_1 - u_1)(v_2 - u_2) \frac{\partial^2 f}{\partial x \partial y}(x,y)
+ R(x,y,v_1 - u_1,v_2 - u_2)
$$

and

$$
k(v_1, v_2) = k(u_1, u_2) + (v_1 - u_1) \frac{\partial k}{\partial x}(u_1, u_2)
+ (v_2 - u_2) \frac{\partial k}{\partial y}(v_1, v_2)
+ \frac{(v_1 - u_1)^2}{2} \frac{\partial^2 k}{\partial x^2}(u_1, u_2)
+ (v_2 - u_2)^2 \frac{\partial^2 k}{\partial y^2}(v_1, v_2).
$$

Upon substituting, cancelling, and rearranging,

$$
0 = \max_{(u_1, u_2) \in \mathbb{R}^2} \left( (v_1 - u_1) \left[ \frac{\partial f}{\partial x}(x,y) - \frac{\partial k}{\partial x}(u_1, u_2) \right]
+ (v_2 - u_2) \left[ \frac{\partial f}{\partial y}(x,y) - \frac{\partial k}{\partial y}(u_1, u_2) \right]
+ \frac{(v_1 - u_1)^2}{2} \left[ \frac{\partial^2 f}{\partial x^2}(x,y) - \frac{\partial^2 k}{\partial x^2}(u_1, u_2) \right]
$$

$$
+ \frac{(v_2 - u_2)^2}{2} \left[ \frac{\partial^2 f}{\partial y^2}(x,y) - \frac{\partial^2 k}{\partial y^2}(v_1, v_2) \right]
+ \frac{(v_1 - u_1)(v_2 - u_2)}{2} \frac{\partial^2 f}{\partial x \partial y}(x,y)\right)
+ R(x,y,v_1 - u_1,v_2 - u_2)
$$

$$
+ \frac{(v_1 - u_1)^2}{2} \frac{\partial^2 k}{\partial x^2}(u_1, u_2)
+ (v_2 - u_2)^2 \frac{\partial^2 k}{\partial y^2}(v_1, v_2)
+ \frac{(v_1 - u_1)(v_2 - u_2)}{2} \frac{\partial^2 f}{\partial x \partial y}(x,y)
$$

$$
+ R(x,y,v_1 - u_1,v_2 - u_2).
$$

Therefore, the bound $B$ is satisfied.
\[
+ \left( \frac{v_2 - u_2}{2} \right)^2 \left[ \frac{\partial^2 f(x, y)}{\partial y^2}(x, y) - \frac{\partial^2 k}{\partial y^2}(v_1, v_1) \right] \\
+ R(x, y, v_1 - u_1, v_2 - u_2)
\]

Upon substituting the values of the partial derivatives of \( k \), unfolding the minimum and maximum operation and reorganizing these results, there exists a \( u_1^0 \) and \( u_2^0, -K \leq u_1^0 \leq K \) and \( -K \leq u_2^0 \leq K \) such that
\[
(u_1^0 - v_1) \left[ \frac{\partial f}{\partial x}(x, y) + C u_1^0 \right] + (u_2^0 - v_2) \left[ \frac{\partial f}{\partial y}(x, y) + C u_2^0 \right] \\
\leq \left( \frac{v_1 - u_1^0}{2} \right)^2 \left[ \frac{\partial^2 f}{\partial x^2}(x, y) + C \right] \\
+ \left( \frac{v_2 - u_2^0}{2} \right)^2 \left[ \frac{\partial^2 f}{\partial y^2}(x, y) + C \right] \\
+ R(x, y, v_1 - u_1^0, v_2 - u_2^0).
\]

But since
\[-KC \leq \frac{\partial f}{\partial x}(x, y) \leq KC\]
for every \( u_1^0 \), there exists a \( v_1^0 \), \( |v_1^0 - u_1^0| = 1 \) and \( -K \leq v_1^0 \leq K \), such that
\[0 \leq (u_1^0 - v_1^0) \left[ \frac{\partial f}{\partial x}(x, y) + C u_1^0 \right] \]
and since
\[-KC \leq \frac{\partial f}{\partial y}(x, y) \leq KC\]
for every \( u_2^0 \), there exists a \( v_2^0 \), \( |v_2^0 - u_2^0| = 1 \) and \( -K \leq v_2^0 \leq K \), such that
\[0 \leq (u_2^0 - v_2^0) \left[ \frac{\partial f}{\partial y}(x, y) + C u_2^0 \right].\]

Hence,
\[0 \leq \frac{1}{2} \left[ \frac{\partial^2 f}{\partial x^2}(x, y) + C \right] + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial y^2}(x, y) + C \right] \\
+ R(x, y, v_1^0 - u_1^0, v_2^0 - u_2^0).
\]

Finally, since \( B \geq |R(x, y, v_1^0 - u_1^0, v_2^0 - u_2^0)| \),
\[-2B - 2C \leq \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y).
\]

References


Figure 1 illustrates the extraction of the body and handle of a shape \( F \) by opening with a disk for the body and taking the residue of the opening for the handle.
### Figure 2

Figure 2 illustrates the calculations for a greyscale dilation.

### Figure 3

Figure 3 illustrates the calculations for a greyscale erosion.