INTERPRETATION OF OPTICAL FLOW
BY ROTATION DECOUPLING

A. Mitiche1, X. Zhuang2, and R Haralick3

1 Institut National de la Recherche Scientifique
3. Place du Commerce, Ile-des-Sours, P.Q., H3E 1H6, Canada
2 University of Washington, Seattle, WA 98195

Abstract. This study deals with the problem of recovering structure and motion in space using optical flow. When the rotational component is decoupled from the optical flow, the resulting vector field is such that all the lines supporting the vectors intersect at the focus of expansion. This rather simple observation leads to, and explains to a certain extent, three otherwise seemingly totally different formulations: the depth-free nonlinear formulation, the linear formulation supported by search, and the linear formulation. The goal of this paper is twofold: describe the derivation of the formulations from rotation decoupling, and provide a statement on their effectiveness, advantages, and disadvantages.

1. Introduction

First, let us briefly examine the few basic equations from which all studies start. Refer to Figure 1. The viewing system is modelled by a central projection model [1]. For simplicity, and without loss of generality, the focal length is 1 unit of measurement. A point \( P \) in space has coordinates \( (X, Y, Z) \) in \( S \) and image coordinates \( (x, y) \). All these positional variables are functions of time. The projective relations can be simply written as:

\[
x = X/Z ; \quad y = Y/Z
\]

The velocity \( P' \) of the motion of object point \( P \), relative to \( S \), is described in terms of a rotational velocity \( \Omega = (\omega_1, \omega_2, \omega_3) \) and a translational velocity \( T = (t_1, t_2, t_3) \) as \( P' = \Omega \times OP \) (assuming a moving object and a stationary viewing system). Differentiating Equation (1) with respect to time and after some algebraic manipulations we can obtain the expression of optical velocity \( (u, v) \) at \((x, y)\):

\[
u = -z \omega_1 + (1 + x^2) \omega_2 - y \omega_3 + (t_1 - xt_3)/Z
\]

\[
v = -(1 + y^2) \omega_1 + xy \omega_2 + x \omega_3 + t_2 - yt_3)/Z
\]

The optical velocity can be written as the sum of two terms.

\[
u = u^T + u^R \quad ; \quad v = v^T + v^R
\]

\[
u^T = (t_1 - xt_3)/Z \quad ; \quad u^R = -z \omega_1 + (1 + x^2) \omega_2 - y \omega_3 \quad (4a)
\]

\[
v^T = (t_2 - yt_3)/Z \quad ; \quad v^R = -(1 + y^2) \omega_1 + xy \omega_2 + x \omega_3 \quad (4b)
\]

The examination of Equations (2) quickly reveals that one of the main difficulties of the problem resides in the occurrence of depth, \( Z \), in these equations. For one thing, the presence of \( Z \) makes the equations non-linear. Also, a direct method would call for the observation of (at least) five points from which, after setting scale, ten equations in ten unknowns can be written. Five of these unknowns will be the relative depths of the points. Each additional point observed will introduce two new equations and one new unknown which is the depth of this additional point. The system of equations is non-linear, its numerical solution requiring initial guesses to the solution. Non-linear systems with such a large number of unknowns and equations are known to be numerically unstable unless a rather close approximation to the solution is available. For the problem at hand the availability of such good approximations is most often impractical to assume. Most studies have therefore sought to develop strategies where the presence of depth is eliminated or its effects weakened. However, this does not imply that the absence of depth or depth effects is the only desirable property of a formulation. The use of inexact vision principles [2], spatial integration (the proper use of large optical flow areas), and multiple sensor integration [3], can all prove to be beneficial.

Some approaches have simply assumed that depth is known or that a mechanism for obtaining depth, such as stereovision, is available [4]-[7]. Others have relied on the existence in the scene of pairs of distinct points on the same projecting line (which have equal rotational component of optical velocity), on a representation of depth as a smooth surface in the other two spatial variables (where depth values in a neighborhood are replaced by a few parameters that describe the surface, thereby reducing the number of depth unknowns) [8]-[11], on the explicit use of rigidity [12], on schemes based on the minimization of various error functions [13]- [14], or on the hypothesis-test paradigm [15]-[20]. A number of other schemes have been proposed where the complexity of the problem has been lessened by restricting object motions to translations or by restricting object geometry or approximating it by simple surfaces (e.g. planar surfaces) [20]-[24]. Recently, a linear method has been developed [25]. Other approaches have used the additional information provided by multiple views [26]-[27]. The article [28] points at some of the inherent ambiguities in recovering 3D motion and structure from optical flow.

This paper describes rotation decoupling. Rotation decoupling leads to and explains three otherwise seemingly totally different methods of solution: the depth-free nonlinear method, the linear method supported by search, and the linear method. As explained in detail subsequently,
decoupling the rotational component from the optical flow results in a (image plane) vector field due only to the translational component and is, therefore, such that all the lines supporting the vectors intersect at the focus of expansion (FOE). Although based on this property of rotational motion, the approach, however, treats the case of general motion. Although other methods have considered, directly or indirectly, such a property, rotation decoupling is different, however, in the way the property is exploited. As well, computations are not restricted to special points in the image such as points along occluding boundaries or points aligned along projecting rays and no property of object shape, other than rigidity, is assumed.

2. Rotation decoupling

If we subtract the rotational component from the expression of optical velocity in Equations (3) we obtain:

\[ u - u^R = u^T \tag{5a} \]

\[ v - v^R = v^T \tag{5b} \]

Taking the ratio of (5a) and (5b) we can write:

\[ \frac{(v - v^R)}{(u - u^R)} = \frac{v^T}{u^T} \tag{6} \]

The ratio in Equation (6) is simply the slope of the line supporting the vector \((u,v)\) at \((x,y)\), i.e. the line through \((\mathbf{x},y)\) and the FOE \((x_0,y_0)\). Therefore:

\[ \frac{(v - v^R)}{(u - u^R)} = \frac{(y - y_0)}{(x - x_0)} \tag{7} \]

Expanding \(u^R\) and \(v^R\) and rewriting (7) leads to the basic equation:

\[ (v + (1 + y^2)\omega_1 - xy\omega_2 - x\omega_3)(x - x_0) \]

\[ - (u + xy\omega_1 - (1 + x^2)\omega_2 + y\omega_3)(y - y_0) = 0 \]

\[ \tag{8} \]

Note that depth does not appear in Equation (8) above. The equation is valid at any instant of time when the translation component is not parallel to the image plane, in which case the FOE is at infinity. Also, \((x_0,y_0)\) is undefined for the case of pure rotational motion, but Equation (8) still holds for arbitrary \((x_0,y_0)\) because \(u^T = v^T = 0\), or \(u - u^R = v - v^R = 0\). However, the case of pure rotational motion should be treated separately as indicated in a subsequent section. Now we will examine three different methods for solving the problem on the basis of Equation (8). But before we do that we want to point out that once \((\omega_1,\omega_2,\omega_3)\) and \((x_0,y_0)\) are known, the direction of translation and relative depth can be determined. Indeed the direction of translation is obtained from \((x_0,y_0)\) [29]:

\[ t_1 = x_0t_3 \quad ; \quad t_2 = y_0t_3 \quad \tag{9} \]

where \(t_3\) is non-zero and free. Then one can go back to Equations (2) to compute relative depth. In the following discussions we will refer to \(\omega_1,\omega_2,\omega_3,x_0,y_0\) as the parameters of the formulation to distinguish them from the original unknown of the problem, \(\omega_1,\omega_2,\omega_3,t_1,t_2,t_3,\lambda\).

2.1 The Depth-Free Non-Linear Method

Equation (8) is left as is, i.e., an equation in five unknowns. Therefore one can seek a solution to the problem by observing \(k \geq 5\) points, yielding a system of \(k\) equations in five unknowns:

\[ (v_1 + (1 + y_1^2)\omega_1 - x_1y_1\omega_2 - x_1\omega_3)(x_1 - x_0) \]

\[ - (u_1 + xy_1\omega_1 - (1 + x_1^2)\omega_2 + y_1\omega_3)(y_1 - y_0) = 0 \]

\[ \ldots \]

\[ (v_k + (1 + y_k^2)\omega_1 - x_ky_k\omega_2 - x_k\omega_3)(x_k - x_0) \]

\[ - (u_k + xy_k\omega_1 - (1 + x_k^2)\omega_2 + y_k\omega_3)(y_k - y_0) = 0 \]

\[ \tag{10} \]

Since there are only five unknowns in the system of equations above, independently of the number of points used, the redundancy ratio increases rapidly with \(k\).

2.2 The Linear Method Supported by Search

The examination of basic Equation (8) reveals that if we knew \((x_0,y_0)\) then the equation becomes a linear equation in three unknowns. This linear equation is of the form:

\[ a\omega_1 + bw_2 + c\omega_3 = d \]

\[ \tag{11} \]

where

\[ a = (1 + y^2)x' - xy' \quad ; \quad b = -xy' + (1 + x^2)y' \]

\[ c = -xx' - yy' \quad ; \quad d = uy' - vx' \]

and,

\[ x' = x - x_0 \quad ; \quad y' = y - y_0 \]

One can now think of searching for \((x_0,y_0)\) (note that we still need at least five points although only three unknowns appear in the equations). But the domain of \((x_0,y_0)\) being \(R^2\) (the entire plane), one should not try, of course, to search directly such a space unless something is known about the motion that would confine the search to a known small area. What should be done instead is the following. First recall that the direction of translation can be obtained from \((x_0,y_0)\): \(t_3 = x_0t_3\) and \(t_2 = y_0t_3\), where \(t_3\) is non-zero and free. Since \(t_3\) is free we can take the vector \((t_1,t_2,t_3)\) to be a unit vector. Also note that \((x_0,y_0)\) is the same whether we consider the vector \((t_1,t_2,t_3)\) or its negative \((-t_1,-t_2,-t_3)\). Therefore, to search for \((x_0,y_0)\) we can digitize and search half a sphere. This can be done by searching a two-dimensional range of angles using the formulas:

\[ t_1 = \cos(\phi_1)\sin(\phi_2) \]

\[ t_2 = \sin(\phi_1)\sin(\phi_2) \tag{12} \]

\[ t_3 = \cos(\phi_2) \]

The answer can be taken to be the couple \((x_0,y_0)\) which yields the minimum residual in the system of linear equations. The search method above is similar to the search methods in [16],[19] although we are not, here, searching the 3D space of rotational components, nor do we have the burden of computing line intersections as an error measure is readily available in the residual of linear Equation (11). This search method is also similar to the methods in [15],[17],[18] although the difference here is that depth
is eliminated at the start from the formulation (Equation (8)), search is directly performed (Equation (12)) using the residual of linear equation (11) for the error measure. However, the problem of developing a sophisticated search strategy and error function, although interesting, is marginal to the present study and is not addressed. The design of an appropriate error function has been addressed in 18.

2.3 The Linear Method

After straightforward algebraic manipulations, one can rewrite basic Equation (8) as :

$$-x(\omega_1 + x_0 \omega_3) - y(\omega_2 + y_0 \omega_3) + (x^2 + y^2) \omega_3$$
$$+ vx_0 - vy_0 + (1 + y^2)x_0 \omega_1 - xy(x_0 \omega_2 + y_0 \omega_1)$$
$$+(1 + x^2)y_0 \omega_2 = xv - yv$$

(13)

Letting

$$\alpha_1 = \omega_1 + x_0 \omega_3 \ ; \ \alpha_2 = \omega_2 + y_0 \omega_3$$
$$\alpha_3 = \omega_3 \ ; \ \alpha_4 = x_0 \ ; \ \alpha_5 = y_0$$
$$\alpha_6 = x_0 \omega_1 \ ; \ \alpha_7 = x_0 \omega_2 + y_0 \omega_1$$
$$\alpha_8 = y_0 \omega_2 \ ; \ b = vx - vy$$

Then Equation (13) can be written as a linear equation in eight unknowns.

$$-x \alpha_1 - y \alpha_2 + (x^2 + y^2) \alpha_3 + v \alpha_4 - u \alpha_5$$
$$+(1 + y^2) \alpha_6 - xy \alpha_7 + (1 + x^2) \alpha_8 = b$$

(15)

One can therefore attempt to solve the problem with eight or more point observations. Each observation will yield one equation. In matrix form, the system is written as :

$$A \alpha = B$$

(16)

Where, for eight observations, matrix A is;

$$
\begin{pmatrix}
-x_1 & -y_1 & x_1^2 + y_1^2 & v_1 & -u_1 & 1 + y_1^2 & -x_1 \omega_1 & 1 + x_1^2 \\
n & -y_2 & x_2^2 + y_2^2 & v_2 & -u_2 & 1 + y_2^2 & -x_2 \omega_1 & 1 + x_2^2 \\
n & -y_3 & x_3^2 + y_3^2 & v_3 & -u_3 & 1 + y_3^2 & -x_3 \omega_1 & 1 + x_3^2 \\
n & -y_4 & x_4^2 + y_4^2 & v_4 & -u_4 & 1 + y_4^2 & -x_4 \omega_1 & 1 + x_4^2 \\
n & -y_5 & x_5^2 + y_5^2 & v_5 & -u_5 & 1 + y_5^2 & -x_5 \omega_1 & 1 + x_5^2 \\
n & -y_6 & x_6^2 + y_6^2 & v_6 & -u_6 & 1 + y_6^2 & -x_6 \omega_1 & 1 + x_6^2 \\
n & -y_7 & x_7^2 + y_7^2 & v_7 & -u_7 & 1 + y_7^2 & -x_7 \omega_1 & 1 + x_7^2 \\
n & -y_8 & x_8^2 + y_8^2 & v_8 & -u_8 & 1 + y_8^2 & -x_8 \omega_1 & 1 + x_8^2 \\
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6 \\
\alpha_7 \\
\alpha_8 \\
\end{pmatrix}
= \begin{pmatrix}
b \\
b \\
b \\
b \\
b \\
b \\
b \\
b \\
\end{pmatrix}
$$

Because A is defined in terms of image observables (images positions and optical velocities) we will refer to it conveniently as the observable matrix. Uniqueness of solution can be stated and shown as follows.

Proposition : Let eight points be observed and let A be the corresponding observable matrix. If Det(A) ≠ 0 then the parameters of the formulation are determined uniquely.

Proof : Let there be two solutions (ω_1, ω_2, ω_3, x_0, y_0) and (ω'_1, ω'_2, ω'_3, x'_0, y'_0). These satisfy the basic motion equation (8) for each of the eight observed points. Then vectors α and α' the components of which are defined as in (13) by :

$$\alpha_1 = \omega_1 + x_0 \omega_3 \ ; \ \alpha'_1 = \omega'_1 + x'_0 \omega'_3$$
$$\alpha_2 = \omega_2 + y_0 \omega_3 \ ; \ \alpha'_2 = \omega'_2 + y'_0 \omega'_3$$
$$\alpha_3 = \omega_3 \ ; \ \alpha'_3 = \omega'_3$$
$$\alpha_4 = x_0 \ ; \ \alpha'_4 = x'_0$$
$$\alpha_5 = y_0 \ ; \ \alpha'_5 = y'_0$$
$$\alpha_6 = x_0 \omega_1 \ ; \ \alpha'_6 = x'_0 \omega'_1$$
$$\alpha_7 = x_0 \omega_2 + y_0 \omega_1 \ ; \ \alpha'_7 = x'_0 \omega'_2 + y'_0 \omega'_1$$
$$\alpha_8 = y_0 \omega_2 \ ; \ \alpha'_8 = y'_0 \omega'_2$$

are solutions to the linear system of equations (16). Because Det(A) ≠ 0 these two solutions must be equal, i.e.,

$$\alpha_i = \alpha'_i \text{ for } i = 1, \ldots, 8.$$ Then

$$\alpha_3 = \alpha'_3 \Rightarrow \omega_3 = \omega'_3$$
$$\alpha_4 = \alpha'_4 \Rightarrow x_0 = x'_0$$
$$\alpha_5 = \alpha'_5 \Rightarrow y_0 = y'_0$$
$$\alpha_6 = \alpha'_6 \wedge \omega_3 = \omega'_3 \wedge x_0 = x'_0 \Rightarrow \omega_1 = \omega'_1$$
$$\alpha_7 = \alpha'_7 \wedge \omega_3 = \omega'_3 \wedge y_0 = y'_0 \Rightarrow \omega_2 = \omega'_2$$

Therefore the two solutions are equal, which completes the proof. The unique solution satisfies (14) and can therefore be computed from (14):

$$\omega_3 = \omega_3 \ ; \ x_0 = x_0 \ ; \ y_0 = y_0$$
$$\omega_1 = \alpha_1 - x_0 \omega_3 \ ; \ \omega_2 = \alpha_2 - y_0 \omega_3$$

(17)

Note. One can rewrite Equation (16) as :

$$(A, -B) \beta = 0$$

Where

$$\beta = (\beta_1, \ldots, \beta_8)^T = (\alpha_1, \ldots, \alpha_8, 1)^T$$

If the rows of matrix (A, −B) are called B_i then Equation (16) is equivalent to the system :

$$B_i \beta = 0 \quad i = 1, \ldots, n$$

Its least-squares formulation is :

$$\min_{||\beta||=1} \beta^T \sum_{i=1}^n (B_i^T B_i) \beta$$

This is a least-squares formulation as in [25] although the coefficient matrices are slightly different.

2.4 The Case of Pure Rotational Motion

Suppose now that the motion, at the time of observation, is a pure rotation such that the analysis described in the preceding section cannot be applied. For such a motion Equations (2a) and (2b) become :

$$u = -x y \omega_1 + (1 + x^2) \omega_2 - y \omega_3$$

(18a)

$$v = -(1 + y^2) \omega_1 + x y \omega_2 + x \omega_3$$

(18b)

These are linear equations in the three unknowns ω_1, ω_2, and ω_3. Theoretically, the image coordinates and optical velocity of each point in a set of observed space points satisfies Equation (18). Therefore, the observation of at least two points would confirm the existence of a pure rotational motion (or of a motion which masquerades a pure rotational motion). But, of course, one should expect Equation (18) to be satisfied only approximately because of the various data perturbations and uncertainties (sensor noise, geometric distortions, algorithmic uncertainties associated with image point location and optical velocity values, numerical evaluations, etc.). As well, a general motion may appear to be a rotational motion (up to the uncertainties mentioned) when the contribution of the translational component to optical flow is small in comparison to that of the rotational component.
3. Simulation and discussion

In the simulation, the non-linear system of equations has been solved using the routine ZXSSQ under the IMSL library. The linear equations of the other two methods have been solved by linear least squares. The focal length is 1cm. Table 1 contains the image coordinates and depth (Z-value) of all the points used from which data is taken. For points at such distances and with a 1cm focal length the pixel size of an equivalent central projection model is approximately $10^{-3}$ cm [1].

Results for the non-linear method are shown in Tables 2a-2d. The effect of noisy image positions when five points are used is indicated in Table 2a. The initial guess to the solution is random and within 50% of the actual solution. Although the method is robust to noise in image positions, it is more sensitive to noise in optical velocities (Table 2b). Using a better initial approximation improves the computation substantially (Table 2c) but 5% noise in optical velocities overwhelms the calculation of the FOE. One has to go to a much higher number of points to recover (Table 3d).

Results for the linear method supported by search are in Tables 3a-3d. When a small number of points is used multiple solutions are likely to occur. Therefore, the occurrence of couples $(x_0, y_0)$ comparably good numerically is likely during the search; unless something is known about the solution, there is no reason to prefer one such couple over another. In fact, if one uses the smallest residual to select the answer, then a change in the resolution of the search space might yield different answers (Table 3a). This behaviour is less likely to occur with a larger number of points (Table 3b). The high noise level used (Table 3b) indicates that the method is robust to noise in image positions. It is however more sensitive to optical flow noise (Table 3c). Going to a higher number of points did not affect the results significantly (Table 3d).

Finally, Tables 4a-4c contain the results for the linear method. The method with eight points is robust to noise in image positions (Table 4a), reasonably sensitive to optical flow noise (Table 4b). Using a larger number of points improves the results (Table 4c).

The example we have given is one among several that we experimented with where we have tried to avoid the most obvious inherent pitfalls pointed at in [28]. We have nevertheless experienced with some of the conditions which make the recovery of 3D parameters inherently unstable regardless of which algorithm is used. The following discussion summarizes the main observations. The simulation in this study supports the following conclusions. With fixed imaging geometry and object configuration, the most common pitfall we have found affects mainly the computation of the translation parameters (or focus of expansion).

- The computation of the direction of translation is not as accurate when the absolute value of the translation is small relative to the viewing distance.
- The computation of the direction of translation is not as accurate if the FOE is far from the image points. The limiting case of this condition is a direction of translation parallel to the image plane.
- The use of points with similar optical velocities (such as points close to each other) does not add to the accuracy of computation.

The above observations are true not only for pure translation but for the translational component of general motion as well.

- In the case of general motion, the computation of the translation parameters is not as accurate when the contribution of this component to the optical flow is small relative to that of the rotational component.
- Overdetermination is not always the answer to the occurrence of 'bad' points (points at which unstable behaviour appears). Indeed, it may be the case that the effect of one such point can overwhelm the effect of all other points.
- The above observations is an indication that a sophisticated computational treatment will be needed for any successful and general process that recovers 3D information from optical flow.

Aside from their common behaviour above, how do the three methods compare? Since they all originate from the same basic fact (recall this fact: when the rotational component is subtracted from optical flow, the resulting vector field is due to the translational component only and is, therefore, such that all the lines supporting the vectors intersect at the focus of expansion), one should expect the performance of these methods to be comparable, albeit some conditions are met. Indeed, given a fair initial approximation of the unknowns for the depth-free nonlinear method, and a fine search space for the linear method supported by search, the performance of these methods is comparable to that of the linear method. After all, one can argue that the linear formulation is the result of linearizing a nonlinear formulation by making a change of variables, and the linear formulation supported by search is the result of targeting some of the unknowns for search. Although the nonlinear method and the linear method supported by search do require some conditions for adequate performance (fair initial approximation, fine search space) whereas the linear method does not, the linear method, however, requires a higher minimum number of observations. Requiring a larger number of points can be a disadvantage when reliable optical flow data is sparse.

The nonlinear method would be suitable for tracking from (approximately) known initial position. The linear method supported by search is attractive because it requires a small number of observations and because the basic operation at each search point is the resolution of a small linear system of equations. The effort for this method should be concentrated on developing an efficient and reliable search procedure. The linear method is suitable when a larger number of observations is available.

4. Summary

This paper has described rotation decoupling which consists of decoupling the rotational component from the optical velocity and expressing the fact that the resulting vector field is such that all the lines supporting its vectors intersect at the focus of expansion. This led to, and explained some aspects of the behaviour of, three otherwise seemingly different formulations: the depth-free nonlinear formulation, the linear formulation supported by search, and the linear formulation.
References


27. A. Mitiche, 'Interpretation of Space from Optical Flow Correspondence,' Institut National de la Recherche Scientifique TR 86-11, March 86.


Fig. 1 Viewing System Configuration.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.1</td>
<td>1.1</td>
<td>303.2</td>
</tr>
<tr>
<td>-0.1</td>
<td>1.1</td>
<td>339.2</td>
</tr>
<tr>
<td>-1.1</td>
<td>0.1</td>
<td>338.4</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>209.9</td>
</tr>
<tr>
<td>0.7</td>
<td>1.1</td>
<td>308.0</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9</td>
<td>336.7</td>
</tr>
<tr>
<td>0.3</td>
<td>1.1</td>
<td>359.2</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.7</td>
<td>363.2</td>
</tr>
<tr>
<td>-0.7</td>
<td>0.7</td>
<td>257.9</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.7</td>
<td>383.7</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7</td>
<td>337.6</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>257.9</td>
</tr>
<tr>
<td>-1.1</td>
<td>-0.7</td>
<td>276.2</td>
</tr>
<tr>
<td>-0.7</td>
<td>-0.7</td>
<td>377.4</td>
</tr>
<tr>
<td>-0.1</td>
<td>-0.7</td>
<td>218.4</td>
</tr>
</tbody>
</table>

Table 1 Image coordinates and depth (Z) of all the data points used in the experiment. The unit of measurement is the centimeter (cm).

<table>
<thead>
<tr>
<th>Noise</th>
<th>ω₁</th>
<th>ω₂</th>
<th>ω₃</th>
<th>x₀</th>
<th>y₀</th>
<th>z₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>1</td>
<td>0.00999</td>
<td>0.01000</td>
<td>0.01001</td>
<td>0.698</td>
<td>0.498</td>
<td></td>
</tr>
<tr>
<td>4.5</td>
<td>0.00094</td>
<td>0.01001</td>
<td>0.01001</td>
<td>0.687</td>
<td>0.487</td>
<td></td>
</tr>
</tbody>
</table>

Table 2a Nonlinear Method. Effect of noisy image positions. Noise is uniformly distributed between +x pixels and -x pixels, x being shown in column labeled ‘noise’. Five points have been used and the initial guess to each unknown is random and within 50% of the actual value.

<table>
<thead>
<tr>
<th>Noise</th>
<th>ω₁</th>
<th>ω₂</th>
<th>ω₃</th>
<th>x₀</th>
<th>y₀</th>
<th>z₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>5%</td>
<td>0.00997</td>
<td>0.00994</td>
<td>0.01004</td>
<td>0.626</td>
<td>0.472</td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>0.00982</td>
<td>0.01005</td>
<td>0.01004</td>
<td>0.684</td>
<td>0.441</td>
<td></td>
</tr>
</tbody>
</table>

Table 2b Nonlinear Method. Effect of noisy optical velocities. Noise is random with zero mean and within 5% the actual value as shown in the column labeled ‘noise’. Five points are used and the initial guess to each unknown is random and within 50% of the actual value.

<table>
<thead>
<tr>
<th>Noise</th>
<th>ω₁</th>
<th>ω₂</th>
<th>ω₃</th>
<th>x₀</th>
<th>y₀</th>
<th>z₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>1%</td>
<td>0.00997</td>
<td>0.00994</td>
<td>0.01004</td>
<td>0.626</td>
<td>0.472</td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>0.00982</td>
<td>0.01005</td>
<td>0.01004</td>
<td>0.684</td>
<td>0.441</td>
<td></td>
</tr>
</tbody>
</table>

Table 2c Nonlinear Method. Effect of noisy optical velocities. Noise is random with zero mean and within 5% the actual value as shown in the column labeled ‘noise’. Five points are used and the initial approximation to the solution is random and within 20 percent of the actual solution.

<table>
<thead>
<tr>
<th>Points</th>
<th>ω₁</th>
<th>ω₂</th>
<th>ω₃</th>
<th>z₀</th>
<th>y₀</th>
<th>z₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>Actual</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>10</td>
<td>0.01010</td>
<td>0.01275</td>
<td>0.00952</td>
<td>-0.206</td>
<td>0.655</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.01186</td>
<td>0.00906</td>
<td>0.01107</td>
<td>0.514</td>
<td>0.569</td>
<td></td>
</tr>
</tbody>
</table>

Table 2d Nonlinear Method. Effect of number of points. The initial approximation is random and within 20 percent of the actual solution.