GRAYSCALE MORPHOLOGY

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Abstract

Binary image morphology has been in use for a number of years. Recently, the extension of binary morphology to grayscale morphology has been made. This paper summarizes the extension by illustrating the homomorphism which establishes the relation between grayscale morphology and binary morphology. There are a number of open statistical questions which this extension suggests and these are discussed in the conclusion.

I. Introduction

We assume familiarity with the binary morphology. A list of references to the literature is provided in the reference section. In this section we state some of the more important definitions and relationships in binary morphology. The reader should note that our definition for erosion differs slightly from Serra and from the paper by Ripley. Symbols such as $x$, $y$, $z$ are elements of Euclidean $N$-space and symbols such as $A$, $B$, $C$ are subsets of Euclidean $N$-space.

Definitions:

Dilation: $A \ominus B = \{z | z = x + y \text{ where } x \in A \text{ and } y \in B\}$

Translation: $B_x = \{z | z = b + x \text{ for some } b \in B\}$

Reflection: $\overline{B} = \{z | -z \in B\}$

Erosion: $A \circ B = \{z | B \subseteq A\}$

Opening: $A \circ B = (A \ominus B) \circ B$

Closing: $A \bullet B = (A \circ B) \circ B$

Properties:

Commutivity: $A \ominus B = B \ominus A$

Dilation Chain Rule: $(A \ominus B) \ominus C = A \ominus (B \ominus C)$

Erosion Chain Rule: $(A \circ B) \circ C = A \circ (B \circ C)$

Dilation Erosion Duality: $A \ominus B = (A^c \ominus B)^c$

Opening Idempotency: $(A \circ B) \circ B = A \circ B$

Closing Idempotency: $(A \bullet B) \bullet B = A \bullet B$

Opening Antiextensivity: $A \circ B \subseteq A$

Closing Extensivity: $A \bullet B \supseteq A$

Opening Increasing: $A \subseteq B$ implies $A \circ K \subseteq B \circ K$

Closing Increasing: $A \subseteq B$ implies $A \bullet K \subseteq B \bullet K$

Opening Closing Duality: $A \circ B = (A^c \bullet B)^c$

In the next section we give the extension of these definitions and properties to grayscale morphology. In section III we discuss several statistical questions which arise from the grayscale morphology.

II. Grayscale Morphology Extension

In this section we develop the extension of binary image morphology to grayscale imagery. Section II.1 introduces the notions of surface and umbra. Section II.2 gives the definitions of grayscale dilation and erosion. It also gives a convenient computational technique for determining the dilation or erosion of an image. Section II.3 establishes the homomorphism from the grayscale morphologic operations to the binary morphologic operations. Section IV.4 uses the homomorphic relation to establish that the properties of the grayscale morphologic operations are similar to the binary morphologic operations.

II.1 Surface and Umbra

We begin with the concepts of surface of a set and the umbra of a surface. Suppose a set $A$ in Euclidean $N$-space is given. We adopt the convention that the first $(N-1)$ coordinates of the $N$-tuples of $A$ constitute the spatial domain of $A$ and the $N^\text{th}$ coordinate is for the surface. For grayscale imagery, $N = 3$. The top or top surface of $A$ is a function defined on the projection of $A$ onto its first $(N-1)$ coordinates. For each $(N-1)$-tuple $x$ the top surface of $A$ at $x$ is the highest value $y$ such that the $N$-tuple $(x, y) \in A$. This is illustrated in Figure 1. If the space we work in is Euclidean, we can express this using the concept of supremum. If the space is discrete, we use the more familiar concept of maximum. Since we choose to suppress the underlying space in what follows, we use maximum throughout. The careful reader will want to translate maximum to
supremum under the appropriate circumstances.

Definition 1: Let \( A \subseteq B^N \) and \( F = \{ z \in B^{N-1} \mid \text{for some } y \in E, (x, y) \in A \} \). The top or top surface of \( A \), denoted by \( T[A] : F \rightarrow E \), is defined by

\[
T[A](z) = \max \{ y \mid (x, y) \in A \}
\]

Figure 1 illustrates the concept of top or top surface of a set.

Definition 2: A set \( A \subseteq B^{N-1} \times E \) is an umbra if and only if \((x, y) \in A\) implies that \((x, z) \in A\) for every \( z \leq y \).

For any function \( f \) defined on some subset \( F \) of Euclidean \((N-1)\)-space the umbra of \( f \) is a set consisting of the surface \( f \) and everything below the surface.

Definition 3: Let \( F \subseteq B^{N-1} \) and \( f : F \rightarrow E \). The umbra of \( f \), denoted by \( U[f] \), \( U[f] \subseteq F \times E \), is defined by

\[
U[f] = \{(x, y) \in F \times E \mid y \leq f(x)\}
\]

Obviously, the umbra of \( f \) is an umbra.

II.2 Grayscale Dilation and Erosion

Having defined the operations of taking a top surface of a set and the umbra of a surface, we can define grayscale dilation. The grayscale dilation of two functions is defined as the surface of the dilation of their umbra.

Definition 4: Let \( F, K \subseteq B^{N-1} \) and \( f : F \rightarrow E \) and \( k : K \rightarrow E \). The dilation of \( f \) by \( k \) is denoted by \( f \star k \), \( f \star k : F \otimes K \rightarrow E \), and is defined by

\[
f \star k = T[U[f] \otimes U[k]]
\]

The definition of grayscale dilation tells us conceptually how to compute the grayscale dilation, but this conceptual way is not a reasonable way to compute it in hardware. The following theorem establishes that grayscale dilation can be accomplished by taking the maximum of a set of sums. Hence, grayscale dilation has the same complexity as convolution. However, instead of doing the summation of products as in convolution, a maximum of sums is performed.

Proposition 5: Let \( f : F \rightarrow E \) and \( k : K \rightarrow E \). Then \( f \star k : F \otimes K \rightarrow E \) can be computed by

\[
(f \star k)(z) = \max_{x \in F \atop z \in K} \{ f(x - z) + k(z) \}
\]

Proof: Suppose \( z = (f \star k)(x) \). Then \( z = T[U[f] \otimes U[k]](x) \). By definition of surface,

\[
z = \max \{ y \mid (x, y) \in [U[f] \otimes U[k]] \}
\]

By definition of dilation,

\[
z = \max \{ a + b \mid \text{for some } u \in K \text{ satisfying } z - u \in F, \newline (x - u, a) \in U[f] \text{ and } (u, b) \in U[k] \}
\]

By definition of umbra, the largest \( a \) such that \((z - u, a) \in U[f]\) is \( a = f(x - u) \). Likewise, the largest \( b \) such that \((u, b) \in U[k]\) is \( b = k(u) \). Hence

\[
z = \max \{ f(x - u) + k(u) \mid u \in K, (x - u) \in F \}
\]

\[
= \max \{ f(x - u) + k(u) \mid (x - u) \in F \}
\]

The definition for grayscale erosion proceeds in a similar way to the definition of grayscale dilation. The grayscale erosion of one function by another is the surface of the binary erosions of the umbra of one with the umbra of the other.

Definition 6: Let \( F \subseteq B^{N-1} \) and \( K \subseteq B^{N-1} \). Let \( f : F \rightarrow E \) and \( k : K \rightarrow E \). The erosion of \( f \) by \( k \) is denoted by \( f \circ k \), \( f \circ k : F \odot K \rightarrow E \), and is defined by

\[
f \circ k = T[U[f] \odot U[k]]
\]

Evaluating a grayscale erosion is accomplished by taking the minimum of a set of differences. Hence its complexity is the same as dilation. Its form is like correlation with the summation of correlation replaced by the minimum operation and the product of correlation replaced by a subtraction operation. If the underlying space is Euclidean, substitute infimum for minimum.

Proposition 7: Let \( f : F \rightarrow E \) and \( k : K \rightarrow E \). Then \( f \circ k : F \odot K \rightarrow E \) can be computed by

\[
(f \circ k)(x) = \min_{z \in R} \{ f(x + z) - k(z) \}
\]
Proof: Suppose $z = (f \ominus k)(x)$. Then, $z = T[U[f] \ominus U[k]](x)$. By definition of surface, $z = \max \{y \mid (x, y) \in U[f] \ominus U[k]\}$. By definition of erosion

$z = \max\{g \mid \text{for every } (u, v) \in U[k], (x, y) + (u, v) \in U[f]\}$

By definition of umbra,

$z = \max\{y \mid \text{for every } u \in K, y + v \leq f(x + u)\}$

$= \max\{y \mid \text{for every } u \in K, y \leq f(x + u) - v\}$

But $y \leq f(x + u) - v$ for every $v \leq k(u)$ implies $y \leq f(x + u) - k(u)$. Hence,

$z = \max \{y \mid \text{for every } u \in K, y \leq f(x + u) - k(u)\}$

But $y \leq f(x + u) - k(u)$ for every $u \in K$ implies

$y \leq \min_{u \in K}[f(x + u) - k(u)]$.

Now,

$z = \max \{y \mid y \leq \min_{u \in K}[f(x + u) - k(u)]\}$

$= \min_{u \in K}[f(x + u) - k(u)]$

Figure 2 illustrates an example of grayscale dilation and erosion.

Figure 2 shows a woman’s face in the image form and in a perspective projection surface plot form. This image is morphologically processed with a paraboloid structuring element given by $6(8 - r^2 - c^2)$, $-2 \leq r \leq 2$, $-2 \leq c \leq 2$. Figure 2b shows the erosion of the girl’s face in image form and perspective projection surface plot form. Figure 2c shows the dilation of the girl’s face in image form and perspective projection surface plot form.
II.3 The Homomorphism From Grayscale Morphology to Binary Morphology

The basic relationship between the surface and umbra operations is that they are, in a certain sense, inverses of each other. More precisely, the surface operation will always undo the umbra operation. That is, the surface operation is an inverse to the umbra operation. However the umbra operation is not an inverse to the surface operation. Without any constraints on the set \( A \), the strongest statement which can be made is that the umbra of the surface of \( A \) contains \( A \). The following properties are easily proven:

\[
T[U[f]] = f \\
A \subseteq U[T[A]]
\]

**Figure 3 illustrates the umbra of the top surface of a set.**

When the set \( A \) is an umbra, then the umbra of the surface of \( A \) is itself \( A \). In this case the umbra operation is an inverse to the surface operation. That is, if \( A \) is an umbra, then \( A = U[T[A]] \).

Having established that the surface operation is always an inverse to the umbra operation and that the umbra operation is the inverse to the surface operation when the set being operated on itself is an umbra, we are almost ready to develop the umbra homomorphism theorem. First we need to recognize that the dilation of one umbra by another is an umbra and that the erosion of one umbra by another is also an umbra.

**Proposition 8:** Suppose \( A \) and \( B \) are umbras. Then \( A \oplus B \) and \( A \odot B \) are umbras.

The proof of proposition 8 is rather direct.

Now we are ready for the umbra homomorphism theorem which states that the operation of taking an umbra is a homomorphism from the grayscale morphology to the binary morphology.

**Umbra Homomorphism Theorem 9:** Let \( F, K \subseteq E^{N-1} \) and \( f : F \to E \) and \( k : K \to E \). Then

1. \( U[f \oplus k] = U[f] \oplus U[k] \)
2. \( U[f \odot k] = U[f] \odot U[k] \)

**Proof:**

1. \( f \oplus k = T[U[f] \odot U[k]] \) so that \( U[f \oplus k] = U[T[U[f] \odot U[k]]] \). But \( U[f] \odot U[k] \) is an umbra and for sets which are umbra the umbra operation undoes the surface operation. Hence \( U[f \oplus k] = U[T[U[f] \odot U[k]]] = U[f] \odot U[k] \).

2. \( f \odot k = T[U[f] \odot U[k]] \) so that \( U[f \odot k] = U[T[U[f] \odot U[k]]] \). But \( U[f] \odot U[k] \) is an umbra and for sets which are umbra, the umbra operation undoes the surface operation. Hence,

\[
U[f \odot k] = U[T[U[f] \odot U[k]]] = U[f] \odot U[k]
\]

II.4 Properties of Grayscale Morphology

We illustrate how the umbra homomorphism property is used to prove grayscale morphology relationships. First the relationship is wrapped by re-expressing it in terms of umbra and surface operations and then it is transformed through the umbra homomorphism. Finally it is unwrapped using the definitions of grayscale dilation and erosion. We use this technique to state and prove the commutivity and associativity of grayscale dilation and the chain rule for grayscale erosion.

**Proposition 10:** \( f \oplus k = k \oplus f \)

**Proof:**

\[
f \oplus k = T[U[f] \odot U[k]] = T[U[k] \odot U[f]] = k \oplus f
\]

**Proposition 11:** \( k_1 \oplus (k_2 \oplus k_3) = (k_1 \oplus k_2) \oplus k_3 \)

**Proof:**

\[
k_1 \oplus (k_2 \oplus k_3) = T[U[k_1] \odot U[k_2 \oplus k_3]] = T[U[k_1] \odot (U[k_2] \odot U[k_3])] = T[(U[k_1] \odot U[k_2]) \odot U[k_3]]
\]
\begin{align*}
&= T[U(k_1 \oplus k_2) \ominus U[k_3]] \\
&= (k_1 \oplus k_2) \ominus k_3
\end{align*}

**Proposition 12:** \((f \ominus k_1) \ominus k_2 = f \ominus (k_1 \ominus k_2)\)

**Proof:**

\[
(f \ominus k_1) \ominus k_2 = T[U[f \ominus k_1] \ominus U[k_2]] \\
= T[U[f] \ominus U[k_1] \ominus U[k_2]] \\
= T[U[f] \ominus (U[k_1] \ominus U[k_2])] \\
= T[U[f] \ominus U[k_1] \ominus k_2)] \\
= f \ominus (k_1 \ominus k_2)
\]

Grayscale opening and closing are defined in an analogous way to opening and closing in the binary morphology and they have similar properties.

**Definition 13:** Let \(f : F \rightarrow E\) and \(k : K \rightarrow E\). The grayscale opening of \(f\) by structuring element \(k\) is denoted by \(f \circ k\) and is defined by \(f \circ k = (f \ominus k) \ominus k\).

**Definition 14:** Let \(f : F \rightarrow E\) and \(k : K \rightarrow E\). The grayscale closing of \(f\) by structuring element \(k\) is denoted by \(f \bullet k\) and is defined by \(f \bullet k = (f \oplus k) \oplus k\).

Figure 4 shows an example of grayscale opening and closing.

To prove the idempotency of grayscale opening and closing, we need the following property relating functions to their umbras.

**Proposition 15:** Let \(f : F \rightarrow E\) and \(g : G \rightarrow E\). Suppose \(F \subseteq G\). Then \(f \leq g\) if and only if \(U[f] \subseteq U[g]\).

**Proof:** Suppose \(f \leq g\). Let \((x, y) \in U[f]\). Then by definition of umbra, \(y \leq f(x)\). But \(x \in F\) and \(F \subseteq G\) so that \(x \in G\). By supposition, \(f(x) \leq g(x)\). Hence, \(y \leq g(x)\). Now by definition of umbra, \((x, y) \in U[g]\).

Suppose \(U[f] \subseteq U[g]\). Let \(y = f(x)\). Certainly, \((x, y) \in U[f]\). But \(U[f] \subseteq U[g]\) so that \((x, y) \in U[g]\).

Now by definition of umbra, \(y \leq g(x)\).

Having this property, we quickly have

**Proposition 16:** \(g \leq f \circ k\) if and only if \(f \geq g \ominus k\)

**Proof:** \(g \leq f \circ k\) if and only if \(U[g] \subseteq U[f \circ k]\). But \(U[f \circ k] = U[f] \ominus U[k]\). Now \(U[g] \subseteq U[f] \ominus U[k]\) if and only if \(U[f] \supseteq U[g] \ominus U[k]\). But \(U[g] \ominus U[k] = U[g] \ominus k\).

Finally, \(U[f] \supseteq U[g] \ominus k\) if and only if \(f \geq g \ominus k\).

Another property which is immediately obvious is that if one set is contained in a second, then the surface of the first will be no higher at each point than the surface of the second.

**Proposition 17:** Let \(A \subseteq E^{N-1} \times E\) and \(D \subseteq E^{N-1} \times E\). Then \(A \subseteq D\) implies \(T[A](x) \leq T[D](x)\).

**Proof:** Let \(z \in E^{N-1}\) be given. Then, since \(A \subseteq D\),

\[
T[A](z) = \max_{(z, s) \in A} z \leq \max_{(z, s) \in D} z = T[D](z).
\]

Figure 4 illustrates the grayscale opening and closing operation. Figure 4a shows the grayscale opening of the girl's face in image form and in perspective projection surface plot form. The structuring element is the paraboloid described in Figure 2. Figure 4b shows the grayscale closing of the girl's face in image form and in perspective projection surface plot form.
From this fact, it quickly follows that the grayscale opening of a function must be no larger than the function at each point in their common domain. This is the grayscale analog to the antitensionivity property of the binary morphology opening.

Proposition 18: \((f \circ k)(x) \leq f(x)\) for every \(x \in F \circ K\)

Proof: 
\[
(f \circ k)(x) = (f \circ k) \circ k = T[(f \circ k) \circ U[k]] = T[(U[f] \circ U[k]) \circ U[k]]
\]

But \((U[f] \circ U[k]) \circ U[k] \subseteq U[f]\), hence by proposition 63,
\[
T[(U[f] \circ U[k]) \circ U[k]](x) \leq T[U[f]](x)
\]
for every \(x \in (F \circ K) \circ K\). Since \(T[U[f]] = f\), \(T[(U[f] \circ U[k]) \circ U[k]](x) \leq f(x)\).

Likewise, the grayscale closing of a function must be no smaller than the function at each point in their common domain. This is the grayscale analog to the extensivity property of the binary morphology closing.

Proposition 19: \((f \circ k)(x) \leq f(x)\) for every \(x \in F\).

Proof: 
\[
(f \circ k)(x) = (f \circ k) \circ k = T[(f \circ k) \circ U[k]] = T[(U[f] \circ U[k]) \circ U[k]]
\]

But \((U[f] \circ U[k]) \circ U[k] \supseteq U[f]\), hence \(T[(U[f] \circ U[k]) \circ U[k]](x) \geq T[U[f]](x)\) for every \(x \in F\). Since \(T[U[f]] = f\), \(f(x) \leq T[(U[f] \circ U[k]) \circ U[k]](x)\).

Now the idempotency property of opening and closing can be proved by the umbra homomorphism theorem.

Proposition 20: \((f \circ k) \circ k = f \circ k\)

Proof:
\[
(f \circ k) \circ k = T[(U[f] \circ U[k]) \circ U[k]] = T[(U[f] \circ U[k]) \circ U[k] \circ U[k]] = T[U[f] \circ U[k] \circ U[k]] = T[U[f] \circ U[k]] = f \circ k
\]

Proposition 21: \((f \circ k) \circ k = f \circ k\)

Proof:
\[
(f \circ k) \circ k = T[(U[f] \circ U[k]) \circ U[k]] = T[(U[f] \circ U[k]) \circ U[k] \circ U[k]] = T[U[f] \circ U[k] \circ U[k]] = T[U[f] \circ U[k]] = f \circ k
\]

There is a geometric interpretation to the grayscale opening and to the grayscale closing in the same manner that there is a geometric meaning to the binary morphological opening and closing. To obtain the opening of \(f\) by a paraboloid structuring element, for example, take the paraboloid, apex up, and slide it under all the surface of \(f\) pushing it hard up against the surface. The apex of the paraboloid may not be able to touch all points of \(f\). For example, if \(f\) has a spike narrower than the paraboloid, the top of the apex may only reach as far as the mouth of the spike. The opening is the surface of the highest points reached by any part of the paraboloid as it slides under all the surface of \(f\). The formal statement of this is given in Proposition 20.

Proposition 22: \(f \circ k = T\left[ \bigcup_{\{y \mid y \in U[x]\} \subseteq U[f]} U[k] \right]\)

Proof:
\[
f \circ k = T[U[f] \circ U[k]] = T[U[f] \circ U[k]] = T[U[f] \circ U[k]]
\]

We have not mentioned the duality relationship between grayscale dilation and erosion. We need this in order to give the geometric interpretation to closing. Before stating and proving it, we need the definition of grayscale reflection.

Definition 23: Let \(f : F \rightarrow E\). The reflection of \(f\) is denoted by \(f^r : F \rightarrow E\), and is defined by \(f^r(x) = f(-x)\).

Grayscale Dilation Erosion Duality Theorem 24:

Let \(f : F \rightarrow E\) and \(k : K \rightarrow E\). Let \(x \in (F \circ K) \cap (F \circ K)\) be given. Then \((-f \circ k)(x) = ((-f) \circ k)(x)\).

Proof:
\[
(-f \circ k)(x) = \max_{z \in F} [f(x - z) + k(z)] = \min_{z \in F} [-f(x - z) - k(z)] = \max_{z \in K} [-f(x + z) - k(z)] = ((-f) \circ k)(x)
\]
It follows immediately from the grayscale dilation and erosion duality that there is a grayscale opening and closing duality.

Grayscale Opening and Closing Duality Theorem 25:

\[-(f \circ k) = (\neg f) \bullet \tilde{k}\]

Proof:

\[-(f \circ k) = -(f \circ (k \oplus k))
\quad = -(f \circ k) \oplus k
\quad = ((\neg f) \oplus k) \oplus k
\quad = -(\neg f) \bullet \tilde{k}\]

Having the grayscale opening and closing duality, we immediately have \( f \circ k = -(\neg f) \circ \tilde{k} \). In essence, this means that we can think of closing like opening. To close \( f \) with a paraboloid structuring element, we take the reflection of the paraboloid in the sense of Definition 23, turn it upside down (apex down), and slide it all over the top of the surface of \( f \). The closing is the surface of all the lowest points reached by the sliding paraboloid.

III. Statistical Questions

There are a whole host of statistical questions which arise from the operations of grayscale morphology and whose answers have basic importance to image processing. These questions are the morphologic analog to the questions which have long ago been answered in linear filtering theory and regression analysis. The morphologic questions unfortunately, are more difficult to answer because of the inherent nonlinearity of the morphologic operations.

Suppose a grayscale image \( I \) is observed to be composed of an underlying ideal image \( f \) distorted by some noise. From the perspective of morphology, what is the natural statistical noise model for the distortion? What is the natural statistical model for the underlying ideal image \( I \)? Given the statistical models for the underlying ideal image and the noise distortion, what is the optimal morphologic operation to perform on the noisy observed image \( J \) to recover the ideal image \( I \)? That is, what is the best morphological estimate of \( I \)?

What are appropriate morphological estimates of derivatives given the underlying statistical model of the ideal image and the noise? What are the distributions of the best estimates? What constitutes a good hypothesis test of a morphologically estimated derivative to be zero?

IV. Conclusion

We have shown the relationship between binary morphology and grayscale morphology and have suggested a number of statistical estimation problems related to grayscale morphology. Hopefully, this will stimulate some statisticians to think about these problems and advance the statistical field into grayscale morphology.

V. References:


