A Brief Survey of the Relationships Between Finite Random Sets and Morphology

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Abstract

In order to be able to optimally design morphological shape extraction algorithms operating on binary digital images, there needs to be a probability theory for finite random sets and probability relations that show how the probability changes as a finite random set is propagated through a morphological operation. In this paper, we develop such a theory for finite random sets. We then demonstrate how to apply this theory for calculating the probability that a set $S$ perturbed by min or max noise $\mathcal{N}$ and dilated or eroded by a structuring element $K$ is a subset, superset, or hits a given set $R$. In some cases we obtain exact results and in some cases we obtain bounds for the desired probability.

1 Introduction

The tools of mathematical morphology constitute a useful set of tools for the purpose of extracting shape and shape information from image data. However, most morphological algorithms are designed by experience and trial and error methods rather than by an engineering design methodology. The key feature of an engineering design methodology is that once the perturbations of the ideal input are given, the methodology must permit us to determine the random perturbation of the output relative to its ideal. And this paper lays a groundwork for doing this.

This point of view is different from the point of view out of which mathematical morphology has arisen. Classically, mathematical morphology constitutes an algebra. The important aspect of the algebra is all the algebraic relations. So for example, Serra (1982) does not treat noise nor regard what is computed as an estimate for any ideal quantity. Recent papers such as those in Serra (1988) or Goutsias and Schonfeld (1991) are similar in this regard.

Recent work in which some statistics or probability is related to mathematical morphology is Dougherty and Haralick (1991), Dougherty and Zhao (1992), Schonfeld and Goutsias (1991a,b), and Loce and Dougherty (1992a,b,c). These papers begin posing the problem of morphological image representation or image restoration when noise is present. They begin to face the questions we pose in this paper. However, in this paper the object is not image representation or restoration. The problem is developing the required relationships that permit the noise random perturbation to propagate through the morphological operations for whatever purpose they may be used. So here, the morphological operations may be used for estimation of some underlying 2D or 3D shape quantity.

The work of this paper is closest to Goutsias (1992) who has explored discrete random sets from the point of view of the capacity function in a similar vein to Matheron’s (1975) work in the continuous case. His results with the discrete generating functional are theoretically interesting but practically non-useable.
2 Random Set Characterization

2.1 Random Sets and Perturbations

In image geometry, a shape is usually represented by a set of points in the universal set $\Omega$. We will use capital letters like $A, B, C, \ldots$ to represent shapes in $\Omega$. Hence, $A \subseteq \Omega, B \subseteq \Omega, C \subseteq \Omega, \ldots$. In our development, we assume that $\Omega$ is a finite set. A random set is different from a set so we use a different notation for it. Random sets will be denoted by calligraphic letters like $A, B, C, \ldots$.

Related to the probability that a random set is a subset of a given set is the probability that a random set is not a subset of a given set. Of course since these are two mutually exclusive and exhaustive events the probabilities must sum to one. To make it easy to talk about the event that a random set is not a subset of a given set, we introduce the concept of hitting.

Definition 1 A set $S$ hits a given set $X$ if it has a non-empty intersection with the given set. We denote that a set $S$ hits a set $X$ by $S \uparrow X$.

Now we can see that if a set $S$ is not a subset of a given set $R$ then it must have a non-empty intersection with the complement of the given set, that is, $S \cap R^c \neq \emptyset$. Hence, either a set is a subset of a given set or the set hits the complement of the given set. Likewise, either a set is a superset of a given set or the complement of the set hits the given set. Therefore we have: \( P(A \subseteq R) + P(A \uparrow R^c) = 1 \). \( P(A \supseteq R) + P(A^c \uparrow R) = 1 \).

On the basis of what we have understood so far, we can determine how to compute the probability of $S \cup \mathcal{N} \subseteq R$ and the probability of $S \cap \mathcal{N} \supseteq R$. We obtain:

\[
(1) \quad P(S \cup \mathcal{N} \subseteq R) = \begin{cases} 
0 & \text{if } S \uparrow R^c \\
\sum_{\mathcal{N} \subseteq R} P(\mathcal{N} = \mathcal{N}) & \text{if } S \subseteq R
\end{cases}
\]

\[
(2) \quad P(S \cap \mathcal{N} \supseteq R) = \begin{cases} 
0 & \text{if } S^c \uparrow R \\
\sum_{\mathcal{N} \supseteq R} P(\mathcal{N} = \mathcal{N}) & \text{if } S \supseteq R
\end{cases}
\]

As the complement of a random set is a random set, the intersection and the union of two random sets result in random sets. It follows directly from our proposition about how to compute the probability function for a random set that

\[
P(\mathcal{X} \cup \mathcal{Y} = A) = \sum_{\mathcal{X}, \mathcal{Y}} P(\mathcal{X} = X, \mathcal{Y} = Y)
\]

and that

\[
P(\mathcal{X} \cap \mathcal{Y} = A) = \sum_{\mathcal{X}, \mathcal{Y}} P(\mathcal{X} = X, \mathcal{Y} = Y)
\]

When two random sets are independent, this means that the joint probability function of the two random sets is the product of the marginal probability functions of each of the random sets. The independence of random sets permits a simplification of the the calculation for the probability that the union of two random sets is a subset of a given set. Likewise there is a simplification for the calculation of the probability of the intersection of two random sets being a superset of a given set. If $\mathcal{X}, \mathcal{Y}$ are independent random sets, then

\[
(1) \quad P(\mathcal{X} \cup \mathcal{Y} \subseteq R) = P(\mathcal{X} \subseteq R)P(\mathcal{Y} \subseteq R)
\]

\[
(2) \quad P(\mathcal{X} \cap \mathcal{Y} \supseteq R) = P(\mathcal{X} \supseteq R)P(\mathcal{Y} \supseteq R).
\]
3 Fundamental Probability Inequalities

There are a number of probability inequalities that come about almost directly from our proposition about how the probability function for a random set being a subset of a given set is computed. The simplest inequality is when there are two given sets, one a subset of the other. The next relation tells us that the probability that a random set is a subset of the smaller fixed set is smaller than the probability that a random set is a subset of the larger fixed set. If \( A \subseteq B \), \( P(S \subseteq A) \leq P(S \subseteq B) \) If \( A \supseteq B \), \( P(S \supseteq A) \leq P(S \supseteq B) \)

There is a simple relation between the probability of a fixed set \( S \) perturbed by min noise \( N \) being a subset of a given set \( R \) and the probability of a fixed set \( S \) perturbed by max noise \( N \) being a subset of a given set \( R \). As stated in the next relation, the first probability must be greater than the second probability. \( P(S \cap N \subseteq R) \geq P(S \cup N \subseteq R) \)

Likewise, there is a simple relation between the probability of a fixed set \( S \) perturbed by max noise \( N \) being a superset of a given set \( R \) and the probability of a fixed set \( S \) perturbed by min noise \( N \) being a superset of a given set \( R \). As stated in the next relation, the first probability must be greater than the second probability. \( P(S \cap N \supseteq R) \leq P(S \cup N \supseteq R) \)

There are some interesting probability inequalities related to the intersection or union of two random sets. As stated by the following proposition, the probability of the intersection of two random sets being a subset of a given set is bounded above by the maximum of the probability that each of them is a subset of the given set. \( P(\mathcal{X} \cap \mathcal{Y} \subseteq R) \geq \max\{P(\mathcal{X} \subseteq R), P(\mathcal{Y} \subseteq R)\} \)

From this it immediately follows that the probability that the intersection of two random sets being a superset of a given set is greater than the maximum of the probability that each of them is a superset of the given set. \( P(\mathcal{X} \cap \mathcal{Y} \supseteq R) \geq \max\{P(\mathcal{X} \supseteq R), P(\mathcal{Y} \supseteq R)\} \)

Likewise, the probability that the intersection of two random sets being a subset of a given set is bounded above by the minimum of the probability that each of them are subsets of the given set. \( P(\mathcal{X} \cup \mathcal{Y} \subseteq R) \leq \min\{P(\mathcal{X} \subseteq R), P(\mathcal{Y} \subseteq R)\} \)

And this relation immediately leads to a relation on the probability of the intersection of two random sets being a superset of a given set. \( P(\mathcal{X} \cap \mathcal{Y} \supseteq R) \leq \min\{P(\mathcal{X} \supseteq R), P(\mathcal{Y} \supseteq R)\} \)

What we would like to do is to determine some bounds on the probabilities for \( (S \cup N) \ominus K \subseteq R \), \( (S \cap N) \ominus K \subseteq R \), \( (S \cup N) \ominus K \subseteq R \), and \( (S \cap N) \ominus K \subseteq R \). We would like these bounds to be only a function of the the random noise perturbation \( N \), the structuring element \( K \), and the set \( R \). To do this we will first work out some morphological set bounds.

4 Morphological Set Bounds

Let \( S, N, K \) denote sets in \( \mathbb{Z}^M \). \( S \) denotes an ideal signal set, \( N \) denotes an outcome noise set, and \( K \) is a structuring element. Each of the morphological set bounds we develop will separate the union/intersection of \( S \) and \( N \) before a morphological operation to a set bounding representation in which the morphological operation is done first on each of \( S \) and \( N \) and then these results are combined in some way with union and intersection operations. Our first two bounds are in fact exact characterizations and are well known in the morphological literature. They are characterizations for dilation under max noise and erosion under min noise.

\[
(S \cup N) \ominus K = (S \ominus K) \cup (N \ominus K)
\]
\[
(S \cap N) \ominus K = (S \ominus K) \cap (N \ominus K)
\]

4.1 Bounds On Erosion Under Max Noise

Set bounds for erosions under max noise or dilation under min noise require some more work. We begin with the set bounds for erosion under max noise. The first two propositions give some relationships which are used in the bound proposition for erosion under max noise. The first proposition gives a characterization for the intersection of a collection of similarly indexed sets in terms of an indexed union of erosions. It basically tells us how to distribute intersections over a union.
Proposition 1 Let \( \Pi = \{(K_1, K_2) \mid K_1 \cap K_2 = \emptyset \text{ and } K_1 \cup K_2 = K\} \) then

\[
\bigcap_{k \in K} (S_k \cup N_k) = (S \ominus K) \cup (N \ominus K)
\]

\[
\bigcup_{(K_1, K_2) \in \Pi \atop K_1 \neq \emptyset, K_2 \neq \emptyset} [(S \ominus K_1) \cap (N \ominus K_2)]
\]

The next proposition tells us in what way erosions produce sets which are smaller than sets produced by dilation.

Proposition 2 If \( K \neq \emptyset \), then \( S \ominus K \subseteq S \ominus \hat{K} \)

The next proposition uses the results of the previous propositions to give the set bounds for erosions under max noise.

Proposition 3 If \( K \neq \emptyset \), then

\[
(S \ominus K) \cup (N \ominus K) \subseteq (S \cup N) \ominus K \\
(S \cup N) \ominus K \subseteq (S \ominus K) \cup (N \ominus K) \cup [(S \ominus K) \cap (N \ominus \hat{K})]
\]

Figure 1 illustrates these bounds.

4.2 Bounds On Dilation Under Min Noise

Using the duality relations, the set bounds on erosion under max noise can be converted to set bounds on dilation under min noise.

If \( K \neq \emptyset \), then

\[
(S \ominus K) \cap (N \ominus K) \cap [(S \ominus \hat{K}) \cup (N \ominus \hat{K})] \subseteq (S \cap N) \ominus K \\
(S \cap N) \ominus K \subseteq (S \ominus K) \cap (N \ominus K)
\]

Figure 2 illustrates these bounds.

5 Probability Bounds For Erosion And Dilation

Now we return to our original problem which was to determine probability characterizations and bounds for erosion under min or max noise and dilation under min or max noise. First we develop characterizations and bounds for erosion under noise and then we develop characterizations and bounds for dilation under noise. Each bound that we develop immediately expands to related bounds due to the duality between dilation and erosion and the relation between subset and hitting. Since there are two kinds of noise: min and max, and since there are two kinds of operations: erosion and dilation, and since there are three kinds of relationships: subset, superset, and hitting, there are a total of twelve possible characterizations or bounds to be developed.

5.1 Probability Bounds For Erosion

In this section we develop a probability characterization for erosion under min noise in a subset relation and bounds for erosion under max noise in a subset relation. The characterizations and bounding relations will each immediately lead to two additional relations.

First we do the characterization.

Proposition 4 If \( S \ominus K \supseteq R \), then \( P(S \cap N) \ominus K \supseteq R) = P(N \ominus K \supseteq R)\)
By the duality relationship between erosion and dilation this characterization immediately leads to a characterization for dilation under max noise in a superset relation. If \( S \oplus K \subseteq R \), then
\[
P\left( (S \cup \mathcal{N}) \oplus K \subseteq R \right) = P\left( \mathcal{N} \oplus K \subseteq R \right)
\]
The relation between subset and hitting leads to a hitting characterization. If \( S \oplus K \subseteq R \), then
\[
P\left( (S \cup \mathcal{N}) \oplus K \uparrow R^c \right) = P\left( \mathcal{N} \oplus K \uparrow R^c \right)
\]Now we can develop the bounds for erosion under max noise in a subset relation. If \( S \oplus K \subseteq R \) and \( K \neq \emptyset \), then
\[
P\left( \mathcal{N} \oplus \bar{K} \subseteq R \right) \leq P\left( (S \cup \mathcal{N}) \ominus K \subseteq R \right) \leq P\left( \mathcal{N} \ominus K \subseteq R \right)
\]
Because of the duality between erosion and dilation the bounds just developed lead to bounds for dilation under max noise in a superset relation.

If \( S \oplus K \supseteq R \) and \( K \neq \emptyset \), then
\[
P\left( \mathcal{N} \ominus \bar{K} \supseteq R \right) \leq P\left( (S \cap \mathcal{N}) \ominus K \supseteq R \right) \leq P\left( \mathcal{N} \ominus K \supseteq R \right)
\]
The relation between subset and hitting leads to the following bounds for erosion under max noise in a hitting relationship. If \( S \oplus K \subseteq R \) and \( \bar{K} \neq \emptyset \), then
\[
P\left( \mathcal{N} \ominus \bar{K} \uparrow R \right) \leq P\left( (S \cup \mathcal{N}) \ominus K \uparrow R \right) \leq P\left( \mathcal{N} \ominus \bar{K} \uparrow R \right)
\]

5.2 Probability Bounds For Dilation

In this section we develop a characterization for the probability of dilation under max noise in a subset relation and then we develop bounds for dilation under min noise in a subset relation. Because of the duality between erosion and dilation this will lead to bounds for erosion under max noise in a superset relation. Because of the relation between subset and hitting the developed bounds lead to bounds for dilation under min noise in a hitting relationship.

If \( S \oplus K \subseteq R \), then
\[
P\left( (S \cup \mathcal{N}) \ominus K \supseteq R \right) = P\left( \mathcal{N} \ominus K \supseteq R - (S \ominus K) \right)
\]
The duality between dilation and erosion immediately leads to the following characterization for erosion under min noise in a subset relation. If \( S \ominus K \subseteq R \), then
\[
P\left( (S \cap \mathcal{N}) \ominus K \subseteq R \right) = P\left( \mathcal{N} \ominus K \subseteq R \cup (S \ominus K) \right)
\]
The relationship between subset and hitting leads to a characterization for erosion under min noise in a hitting relationship. If \( S \ominus K \subseteq R \), then
\[
P\left( (S \cap \mathcal{N}) \ominus \bar{K} \uparrow R^c \right) = P\left( \mathcal{N} \ominus \bar{K} \uparrow (R \cup (S \ominus K))^c \right)
\]

Now we can state the bounds for dilation under min noise in a subset relation. If \( S \oplus K \supseteq R \) and \( K \neq \emptyset \), then
\[
P\left( \mathcal{N} \ominus K \subseteq R \right) \leq P\left( (S \cap \mathcal{N}) \ominus K \subseteq R \right) \leq P\left( \mathcal{N} \ominus \bar{K} \subseteq R \right)
\]
The duality between erosion and dilation leads to the following bound relation for erosion under max noise in a superset relation. If \( S \ominus K \subseteq R \) and \( K \neq \emptyset \), then
\[
P\left( \mathcal{N} \ominus \bar{K} \supseteq R \right) \leq P\left( (S \cup \mathcal{N}) \ominus K \supseteq R \right) \leq P\left( \mathcal{N} \ominus K \supseteq R \right)
\]
The relationship between subset and hitting leads to the following bounds for hitting. If \( S \ominus K \supseteq R^c \) and \( K \neq \emptyset \), then
\[
P\left( \mathcal{N} \ominus \bar{K} \uparrow R \right) \leq P\left( (S \cap \mathcal{N}) \ominus K \uparrow R \right) \leq P\left( \mathcal{N} \ominus \bar{K} \subseteq R \right)
\]

6 Conclusion

We have developed the concept of a finite random set and the probability function of such a set. We have discussed the use of the min or max noise model by which a fixed set \( S \) is randomly perturbed by the union or intersection of a random set \( \mathcal{N} \). The resulting random set is the observed image.

The observed image is processed morphologically to extract, which really means estimate, some primitive part of the underlying but unobserved shape \( S \). To determine optimal morphological algorithms for this purpose, it is required to be able to compute the probability functions of the observed random set after dilation, erosion, opening, or closing. In this paper we have shown how to compute exactly or how to compute bounds for the probability function of a perturbed set under min noise or max noise, under
dilation or erosion, and in a subset relation, superset relation, or hitting relation with a fixed set R. These relations can be extended in the natural way for the opening and closing operations.

Our future research includes how to estimate the probability function of the perturbing noise and how to determine an optimal morphology algorithm using these results.

References


Figure 1: Figure 1 illustrates the bounds on erosion under max noise. Black dots denote binary one pixels. $K$ is a 5-point cross structuring element. (a) signal $S$; (b) correlated noise $N$: The correlation is achieved by closing the noisy image in Figure 1(b) with a $3 \times 2$ rectangular structuring element. (c) $S \cup N$; (d) $(S \cup N) \ominus K$ (e) lower bound $(S \ominus K) \cup (N \ominus K)$; (f) upper bound $(S \ominus K) \cup (N \ominus K) \cup [(S \ominus K) \cap (N \ominus K)]$. 

\(S \ominus K\) and \(N \ominus K\) are the eroded versions of the signals using $K$ as structuring element. The correlation is achieved by closing the noisy image in Figure 1(b) with a $3 \times 2$ rectangular structuring element.
Figure 2: Illustrates the bounds on dilation under min noise. Black dots denote binary one pixels. $K$ is a 5-point cross structuring element. (a) Signal $S$; (b) correlated noise $N$: The correlation is achieved by closing the noisy image in Figure 2(b) with a $3 \times 2$ rectangular structuring element. (c) $S \cap N$; (d) $(S \cap N) \oplus K$ (e) lower bound $(S \oplus K) \cap (N \oplus K) \cap [(S \oplus K) \cup (N \ominus K)]$; (f) upper bound $(S \oplus K) \cap (N \ominus K)$. 