A Geometric Functional for Derivatives Approximation

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Abstract. *We develop on estimation method, for the derivative field of an image based on Bayesian approach which is formulated in a geometric way. The Maximum probability configuration of the derivative field is found by a gradient descent method which leads to a non-linear diffusion type equation with added constraints. The derivatives are assumed to be piecewise smoothe and the Beltrami framework is used in the development of an adaptive smoothing process.*

1 Introduction

It is widely accepted that gradients are of utmost importance in early vision analysis such as image enhancement and edge detection. Several numerical recipes are known for derivatives estimation. All based on fixed square or rectangular neighborhoods of different sizes. This type of estimation does not account for the structure of images and bound to produce errors especially near edges where the estimate on one side of the edge may wrongly influence the estimate on the other side of it. In places where the image is relatively smooth, least square estimates of derivatives computed over large area neighborhoods will give best results (e.g the facet approach [2], see also [1]). But, in places where the underlying image intensity surface is not smooth, and therefore can not be fitted by a small degree bivariate polynomial, the neighborhood should be smaller and rectangular, with the long axis of the rectangle aligned along the orientation of the directional derivative.

From this viewpoint, it is natural to suggest a varying size and shape neighborhood in order to increase both the robustness of the estimate to noise, and its correctness. Calculating directly for each point of the image its optimal neighborhood for gradient estimation is possible but cumbersome. We Therefore propose an alternative approach, which uses a geometry driven diffusion [8] that produces implicitly, and in a sub-pixel accuracy, the desirable effect. We are not concerned, in this approach, with finding an optimal derivative filter but formulate directly a Bayesian reasoning for the derivative functions themselves.

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The paper is organized as follows: In Section 2 we review the Beltrami framework. A Bayesian formulation of the problem, in its linear form, is presented in Section 3. We incorporate, in Section 4, The Beltrami framework in the Bayesian paradigm, and derive partial differential equations (PDEs) by means of the gradient descent method. Preliminary results are presented in Section 5.

2 A Geometric Measure on Embedded Maps

We represent an image as a two-dimensional Riemannian surface embedded in a higher dimensional spatial-feature Riemannian manifold [11,10,3,4,5,13,12]. Let $\sigma^\mu, \quad \mu = 1, 2$, be the local coordinates on the image surface and let $X^i, \quad i = 1, 2, \dots, m$, be the coordinates on the embedding space than the embedding map is given by

$$(X^1(\sigma^1, \sigma^2), X^2(\sigma^1, \sigma^2), \dots, X^m(\sigma^1, \sigma^2)). \tag{1}$$

Riemannian manifolds are manifolds endowed with a bi-linear positive-definite symmetric tensor which is called a *metric*. Denote by $(\Sigma, (g_{\mu\nu}))$ the image manifold and its metric and by $(M, (h_{ij}))$ the space-feature manifold and its corresponding metric. Then the map $\mathbf{X} : \Sigma \to M$ has the following weight [7]

$$E[X^i, g_{\mu\nu}, h_{ij}] = \int d^2\sigma \sqrt{g} g^{\mu\nu} (\partial_\mu X^i)(\partial_\nu X^j) h_{ij}(\mathbf{X}), \tag{2}$$

where the range of indices is $\mu, \nu = 1, 2$, and $i, j = 1, \dots, m = \dim M$, and we use the Einstein summation convention: identical indices that appear one up and one down are summed over. We denote by g the determinant of $(g_{\mu\nu})$ and by $(g^{\mu\nu})$ its inverse. In the above expression $d^2\sigma\sqrt{g}$ is an area element of the image manifold. The rest, i.e. $g^{\mu\nu}(\partial_\mu X^i)(\partial_\nu X^j)h_{ij}(\mathbf{X})$, is a generalization of L_2. It is important to note that this expression (as well as the area element) does not depend on the local coordinates one chooses.

The feature evolves in a geometric way via the gradient descent equations

$$X^i_t \equiv \frac{\partial X^i}{\partial t} = -\frac{1}{2\sqrt{g}} h^{il} \frac{\delta E}{\delta X^l}. \tag{3}$$

Note that we used our freedom to multiply the Euler-Lagrange equations by a strictly positive function and a positive definite matrix. This factor is the simplest one that does not change the minimization solution while giving a reparameterization invariant expression. This choice guarantees that the flow is geometric and does not depend on the parameterization.

Given that the embedding space is Euclidean, The variational derivative of E with respect to the coordinate functions is given by

$$-\frac{1}{2\sqrt{g}} h^{il} \frac{\delta F}{\delta X^l} = \Delta_g X^i = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu X^i), \tag{4}$$

where the operator that is acting on X^i in the first term is the natural generalization of the Laplacian from flat spaces to manifolds and is called *the second order differential parameter of Beltrami* [6], or in short *Beltrami operator*.

3 Bayesian formulation for derivatives estimate

Denote by (x_r, y_s) the sampling points and by $I^0_{rs} \equiv I^0(x_r, y_s)$ the grey-levels at the sampling points.

From the data i.e. $(x_r, y_s), I^0_{rs}$ we want to infer the underlying function $I(x, y)$ and its gradient vector field $\mathbf{V}(x, y)$. The analysis is easier in the continuum and we refer from now on to I^0 as to a continuous function. In practice we can skip a stage and find the derivatives without referring to the underlying function. The inference is described by the posterior probability distribution

$$P(\mathbf{I}(x, y), \mathbf{V}(x, y)|I^0(x, y)) = \frac{P(I^0(x, y)|\mathbf{I}(x, y), \mathbf{V}(x, y))P(\mathbf{I}(x, y), \mathbf{V}(x, y))}{P(I^0(x, y))}$$

In the numerator the first term $P(I^0(x, y))|\mathbf{V}(x, y))$ is the probability of the sampled grey-level values given the vector field $\mathbf{V}(x, y)$ and the second term is the prior distribution on vector fields assumed by our model. The denominator is independent of $\mathbf{V}$ and will be ignored from now on.

Assuming that $P(A|B)$ is given by a Gibbsian form :

$$P(A|B) = Ce^{-\alpha E(A,B)},$$

we get

$$-\log P(\mathbf{V}(x, y)|I^0(x, y)) = \alpha E_1(I^0(x, y), \mathbf{V}(x, y)) + \beta E_2(\mathbf{V}(x, y)).$$

If we use the Euclidean L_2 norm we get

$$\begin{aligned} E_1(I^0(x, y), \mathbf{V}(x, y)) &= \frac{1}{2}C_1 \int dxdy \left(|\mathbf{V} - \nabla I|^2\right) \\ E_2(\mathbf{V}(x, y)) &= \frac{1}{2}C_2 \int dxdy \left(|\nabla \mathbf{V}|^2\right) + E_3, \end{aligned} \tag{5}$$

where the first term is a fidelity term that forces the vector field $\mathbf{V}$ to be close enough to the gradient vector field of $I(x, y)$. The second term intoduces regularization that guarantees certain smoothness properties of the solution. The second term in E_2 constraints the vector field to be a gradient of a function. Its form is:

$$E_3(I(x, y), \mathbf{V}(x, y) = \frac{1}{2}C_3 \int dxdy(\epsilon^{\mu\nu}\partial_\mu V_\nu)^2 = \frac{1}{2}C_3 \int dxdy(V1_y - V2_x)^2,$$

where $\epsilon^{\mu\nu}$ is the antisymmetric tensor.

Alternatively we may adopt a more sophisticated regularization based on geometric ideas. These are treated in the next section.

Maximization of the posterior probability amounts to the minimization of the energy. We do that by means of the gradient descent method which leads eventually to non-linear diffusion type equations.

4 Derivatives Estimation: Geometric Method

In this section we incorporate the Beltrami framework into the Byesian paradigm. We consider the intensity to be part of the feature space, and the fifth-dimensional embedding map is

$$(X^1 = x, X^2 = y, X^3 = I(x, y), X^4 = V_1(x, y), X^5 = V_2(x, y)). \tag{6}$$

Again we assume that these are Cartesian coordinates of $\mathbb{R}^5$ and therefore $h_{ij} = \delta_{ij}$. That implies the following induced metric:

$$(g_{\mu\nu}(x, y)) = \begin{pmatrix} 1 + I_x^2 + V1_x^2 + V2_x^2 & I_x I_y + V1_x V1_y + V2_x V2_y \\ I_x I_y + V1_x V1_y + V2_x V2_y & 1 + I_y^2 + V1_y^2 + V2_y^2 \end{pmatrix}. \tag{7}$$

The energy functionals have two more terms: The first is a fidelity term of the denoised image with respect to the observed one, and the last is an adaptive smoothing term. The functionals are

$$\begin{aligned} E_0(I(x, y), I^0(x, y)) &= \frac{1}{2} C_0 \int dxdy \sqrt{g} (I - I^0)^2 \\ E_1(I^0(x, y), \mathbf{V}(x, y)) &= \frac{1}{2} C_1 \int dxdy \sqrt{g} \left(|\mathbf{V} - \nabla I^0|^2 \right) \\ E_2(\mathbf{V}(x, y)) &= \frac{1}{2} C_2 \int dxdy \sqrt{g} g^{\mu\nu} (\partial_\mu X^i)(\partial_\nu X^i) \\ E_3(\mathbf{V}(x, y)) &= \frac{1}{2} C_3 \int dxdy \sqrt{g} (\epsilon^{\mu\nu} \partial_\nu V_\mu)^2, \end{aligned} \tag{8}$$

and since the Levi-Civita connection's coefficients are zero, we get the following gradient descent system of equations:

$$\begin{aligned} I_t &= C_2 \Delta_g I - C_0 (I - I^0) \\ V_{\rho t} &= C_2 \Delta_g V_\rho - C_1 (V_\rho - \partial_\rho I^0) + \frac{C_3}{\sqrt{g}} \partial_\rho (\sqrt{g} \epsilon^{\mu\nu} \partial_\nu V_\mu), \end{aligned} \tag{9}$$

with the initial conditions

$$\begin{aligned} I(x, y, t = 0) &= I^0(x, y) \\ V_\rho(x, y, t = 0) &= \partial_\rho I^0(x, y), \end{aligned} \tag{10}$$

where $I^0(x, y)$ is the given image.

It is important to understand that V_1 and V_2 are estimates of I_{0x} and I_{0y} and not of the denoised I_x and I_y.

5 Results and discusssion

The solution of the PDE's was obtained by using the explicit Euler scheme, where the time derivative is forward and the spatial derivatives are central. The stencil was taken as 3×3.

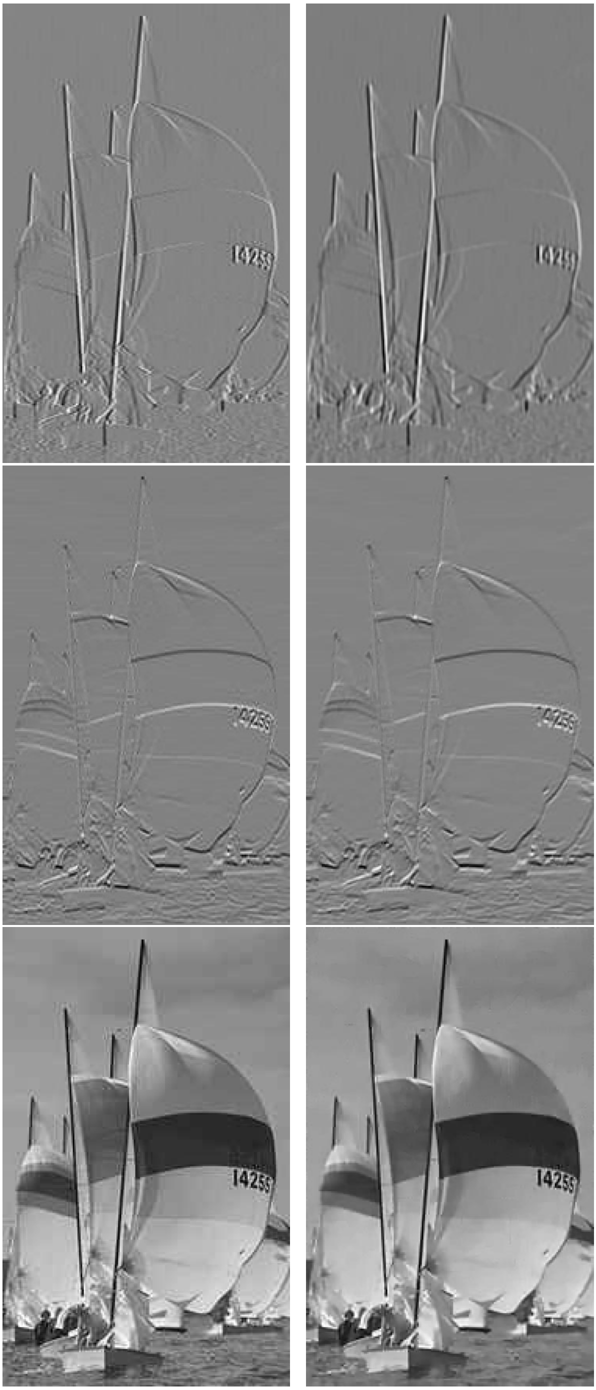

Fig. 1. Upper row, left: The original noisy x derivative. Upper row, right: The x derivative estimation. Middle row, left: The original noisy y derivative. Middle row, right: The y derivative estimation. Lower row, left: The original noisy image. Lower row, right: The denoised image.

We did not optimize any parameter, nor the size of the time steps. For the Euclidean embedding algorithm we chose $C_1 = 0.5$, $C_2 = 1$, $C_3 = 8.5$ and the time step was $\Delta t = 0.005$. The results after 150 iterations are depicted in Fig. (1).

This demonstrates that it is possible to merge Bayesian reasoning and the geometric Beltrami framework in computation of derivative estimations. The requirement that the obtained functions are the x and y derivatives of some underlying function is formulated through a Lagrange multiplier. Close inspection reveals that this requirement is fulfilled only approximately.

An analysis and comparison with statistical based method will appear elsewhere [9].

References

1. M Azaria, I Vitsnudel and Y Y Zeevi "The Design of Tow-Dimensional Gradient estimators Based on One-Dimensional Operators", *IEEE Trans. on Image Processing*, 5, (1996) 155-159.
2. R M Haralick and L G Shapiro *Computer and Robot Vision*, Addison-Wesley Publishing Company, New York, 1992, Chapter 8.
3. R Kimmel, R Malladi and N Sochen, "Images as Embedding Maps and Minimal Surfaces: Movies, Color, Texture, and Volumetric Medical Images", *Proc. of IEEE CVPR'97*, (1997) 350-355.
4. R Kimmel, N Sochen and R Malladi, "On the geometry of texture", Report, Berkeley Labs. UC, LBNL-39640, UC-405, November,1996.
5. R Kimmel, N Sochen and R Malladi, "From High Energy Physics to Low Level Vision", *Lecture Notes In Computer Science:* 1252, First International Conference on Scale-Space Theory in Computer Vision, Springer-Verlag, 1997, 236-247.
6. E Kreyszing, "Differential Geometry", Dover Publications, Inc., New York, 1991.
7. A M Polyakov, "Quantum geometry of bosonic strings", *Physics Letters*, **103B** (1981) 207-210.
8. B M ter Haar Romeny Ed., Geometry Driven Diffusion in Computer Vision, Kluwer Academic Publishers, 1994.
9. N A Sochen, R M Haralick and Y Y Zeevi *in preparation*
10. N Sochen, R Kimmel and R Malladi, "From high energy physics to low level vision", Report, LBNL, UC Berkeley, LBNL 39243, August, Presented in ONR workshop, UCLA, Sept. 5 1996.
11. N Sochen, R Kimmel and R Malladi , "A general framework for low level vision", *IEEE Trans. on Image Processing*, 7, (1998) 310-318.
12. N Sochen and Y Y Zeevi, "Images as manifolds embedded in a spatial-feature non-Euclidean space", November 1998, EE-Technion report no. 1181.
13. N Sochen and Y Y Zeevi, "Representation of colored images by manifolds embedded in higher dimensional non-Euclidean space", IEEE ICIP'98, Chicago, 1998.