Unimodal Search

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Monotonic Increasing Functions

**Definition**

Let $I$ be any interval of the real numbers $\mathbb{R}$. A function $f : I \rightarrow \mathbb{R}$ is **Monotonically Increasing** if and only if for every $(x, y) \in I \times I$, if $x \geq y$, then $f(x) \geq f(y)$. 

![Graph of a monotonically increasing function](image)
Monotonic Decreasing Functions

Definition

Let $I$ be any interval of the real numbers in $\mathbb{R}$. A function $f : I \rightarrow \mathbb{R}$ is **Monotonically Decreasing** if and only if for every $(x, y) \in I \times I$, if $x \geq y$, then $f(x) \leq f(y)$. 
Definition

Let $I$ be any interval of the real numbers in $R$. A function $f : I \rightarrow R$ is **Strictly Increasing** if and only if for every $(x, y) \in I \times I$, if $x > y$, then $f(x) > f(y)$. 
Strictly Decreasing Functions

Definition

Let $I$ be any interval of the real numbers in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is **Strictly Decreasing** if and only if for every $(x, y) \in I \times I$, if $x > y$, then $f(x) < f(y)$. 
Definition

Let \([a, b]\) be any interval of the real numbers in \(R\). A function \(f : [a, b] \rightarrow R\) is **Unimodal** if and only if there exists \(x^* \in [a, b]\) such that

- \(f(x^*) \geq f(x), \ x \in [a, b]\)
- \(f\) is strictly increasing in \([a, x^*]\)
- \(f\) is strictly decreasing in \([x^*, b]\)

Or

- \(f(x^*) \leq f(x), \ x \in [a, b]\)
- \(f\) is strictly decreasing in \([a, x^*]\)
- \(f\) is strictly increasing in \([x^*, b]\)
Unimodal Functions
Suppose $f$ is a unimodal function on $[0, L]$ with a maximum at $x^*$. Suppose $x_1 > x_2$ and $x_1, x_2 \in [0, L]$. Consider $f(x_1)$ and $f(x_2)$. There are 3 cases:

- $f(x_1) < f(x_2)$
- $f(x_1) > f(x_2)$
- $f(x_1) = f(x_2)$
If \( f(x_1) < f(x_2) \), then it is impossible for the maximum to be in the interval \([x_1, L]\). The search need only continue in the interval \([0, x_1]\), an interval of length \(x_1\).
If \( f(x_1) > f(x_2) \), then it is impossible for the maximum to be in the interval \([0, x_2,]\). The search need only continue in the interval \([x_2, L]\), an interval of length \( L - x_2 \).
If \( f(x_1) = f(x_2) \), then it is impossible for the maximum to be in the interval \([0, x_1]\) or \([x_2, L]\). The search need only continue in the interval \([x_1, x_2]\). Without loss of generality, this case can be included either in case 1 or case 2.
Where To Place A Trial

If \( f(x_1) < f(x_2) \), the maximum must be in the interval \([0, x_1]\). If \( f(x_1) > f(x_2) \) the maximum must be in the interval \([x_2, L]\). If either of these intervals were larger than the other, the search could lose efficiency. Therefore

\[ x_1 = L - x_2 \]

The ratio of the length of the new interval of uncertainty to the length of the old interval of uncertainty is

\[ r = \frac{x_1}{L} \]
If $f(x_1) < f(x_2)$, the interval of uncertainty is $[0, x_1]$ and the interior completed trial is $x_2$. We must place the next trial $x_3$ so that

$$x_2 = x_1 - x_3$$

The ratio of the length of the new interval of uncertainty to the length of the old interval of uncertainty is

$$r = \frac{x_2}{x_1}$$
System of Equations

\[ x_1 = L - x_2 \]
\[ r = \frac{x_1}{L} \]
\[ r = \frac{x_2}{x_1} \]

Hence

\[ \frac{x_1}{L} = \frac{x_2}{x_1} \]
\[ x_1^2 - x_2L = 0 \]

Therefore,

\[ x_1 + x_2 = L \]
\[ x_1^2 - x_2L = 0 \]
Unimodal Functions
Golden Search

System of Equations

\[
x_2 = L - x_1
\]
\[
x_1^2 - x_2L = 0
\]

Substituting \( x_2 \) into the second equation,

\[
x_1^2 - (L - x_1)L = 0
\]
\[
x_1^2 + x_1L - L^2 = 0
\]
\[
\left(\frac{x_1}{L}\right)^2 + \left(\frac{x_1}{L}\right) - 1 = 0
\]

\[
\frac{x_1}{L} = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}
\]
\[
= \frac{-1 \pm \sqrt{5}}{2}
\]
Golden Search

\[ \frac{x_1}{L} = \frac{-1 \pm \sqrt{5}}{2} \]

Since, \( \frac{x_1}{L} > 0 \) and \( \sqrt{5} > 1 \)

\[ r = \frac{x_1}{L} = \frac{-1 + \sqrt{5}}{2} \approx .618 \]
Golden Search

\[ r = \frac{-1 + \sqrt{5}}{2} \]

\[ r = \frac{x_1}{L} \]

\[ x_1^2 - x_2L = 0 \]

\[ \left(\frac{x_1}{L}\right)^2 = \frac{x_2}{L} \]

\[ r^2 = \frac{x_2}{L} \]

Diagram:

- 0
- \( r^2L \)
- \( x_2 \)
- \( x_1 \)
- \( L \)
- \( rL \)
Golden Search

If the continued interval of uncertainty is the left interval, then

\[ r^3 L \quad x_3 \]
\[ 0 \quad x_2 \quad x_1 \quad L \]

\[ r^2 L \quad rL \]
Golden Search

If the continued interval of uncertainty is the right interval, then

\[ 0 \quad r^2L \quad x_2 \quad x_1 \quad L \]

\[ rL \quad r^3L \quad x_3 \]
float golden_section_max(float *f, float a, float b, float eps)
{
    r = (-1. + sqrt(5)) / 2;
    x1 = a + r * (b - a);
    x2 = b - r * (b - a);
    while abs(x1 - x2) > eps
    {
        if (f(x1) < f(x2)) /* left interval */
            b = x1;
        else
            a = x2; /* right interval */
        x1 = a + (b - a) * r;
        x2 = b - r * (b - a);
    }
    return (a + b) / 2;
}