Probability Models

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Outline

The Problem

- When there are many variables, the sample size is often too small
- When the sample size is too small, the class conditional joint probability cannot be estimated directly
- There must be some assumptions made to allow low order marginals to be combined in some manner to form class conditional joint probabilities to be used in the classification

The Markov Assumption

$$
\begin{aligned}
p(y_1 \mid y_2 \ldots y_N) &= P(y_1 \mid y_2) \\
p(y_2 \mid y_3 \ldots y_N) &= P(y_2 \mid y_3) \\
&\vdots \\
P(y_{N-2} \mid y_{N-1}, y_N) &= P(y_{N-2} \mid y_{N-1})
\end{aligned}
$$

In general,

$$P(y_n \mid y_{n+1} \ldots y_N) = P(y_n \mid y_{n+1}), n = 1, \ldots N-1$$

Conditional Probability

Now,

$$\begin{aligned} P(x_1 \ldots x_N) &= P(x_1 \,| x_2 \ldots x_N) P(x_2 \ldots x_N) \\ &= P(x_1 \,| x_2 \ldots x_N) P(x_2 \mid x_3 \ldots x_N) P(x_3 \ldots x_N) \end{aligned}$$

Repeating the pattern,

$$P(x_1 \ldots x_N) = \left[\prod_{n=1}^{N-1} P(x_n \,| x_{n+1} \ldots x_N) \right] P(x_N)$$

Under the Markov Assumption

$$P(x_n \mid x_{n+1} \ldots x_N) = P(x_n \mid x_{n+1}), \; n = 1, \ldots N-1$$

Hence,

$$\begin{aligned} P(x_1 \ldots x_N) &= \left[\prod_{n=1}^{N-1} P(x_n \mid x_{n+1} \ldots x_N)\right] P(x_N) \\ &= \left[\prod_{n=1}^{N-1} P(x_n \mid x_{n+1})\right] P(x_N) \end{aligned}$$

The Markov Classifier

Assign $(x_1, \ldots x_N)$ to class c^* when

$$
\begin{aligned}
P(x_1 \ldots x_N \mid c^*) &> P(x_1 \ldots x_N \mid c), c \neq c^* \\
\left[\prod_{n=1}^{N-1} P(x_n | x_{n+1}, c^*)\right] P(x_N | c^*) &> \left[\prod_{n=1}^{N-1} P(x_n | x_{n+1}, c)\right] P(x_N | c)
\end{aligned}
$$

for all other c

The General Markov Classifier

Let $i_1, \ldots, i_N$ be a permutation of $1, \ldots, N$. Assign $(x_1, \ldots x_N)$ to class c^* when

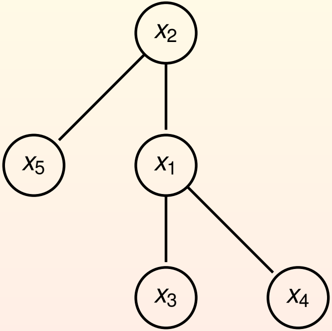
$$\begin{aligned} P(x_1 \ldots x_N \mid c^*) &> P(x_1 \ldots x_N \mid c), c \neq c^* \\ \left[\prod_{n=1}^{N-1} P(x_{i_n} | x_{i_{n+1}}, c^*)\right] P(x_{i_N} | c^*) &> \left[\prod_{n=1}^{N-1} P(x_{i_n} | x_{i_{n+1}}, c)\right] P(x_{i_N} | c) \end{aligned}$$

for all other c

How To Choose the Permutation

- Use the training data to estimate $P(x_i \, | x_j, c)$, $i \neq j$
- For permutation $i_1, \ldots, i_N$
- Use the first half of testing data to estimate the expected gain using $P(x_{i_n} \, | x_{i_{n+1}}, c)$
- Search for the permutation having the largest estimated expected gain
- For the best permutation, get an unbiased estimate of the estimated expected gain using the second half of the testing data

First Order Dependence Trees

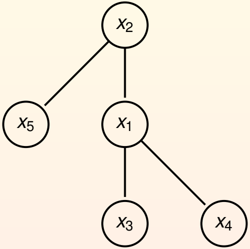


$$P(x_1, x_2, x_3, x_4, x_5) = p(x_1 \mid x_2) P(x_5 \mid x_2) P(x_3 \mid x_1) P(x_4 \mid x_1) P(x_2)$$

First Order Dependence Trees

$$\begin{aligned}
1 &= \sum_{x_1}\sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} P(x_1, x_2, x_3, x_4, x_5) \\
&\sum_{x_1}\sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5} p(x_1 \mid x_2) P(x_5 \mid x_2) P(x_3 \mid x_1) P(x_4 \mid x_1) P(x_2) \\
&\sum_{x_2} P(x_2) \sum_{x_1} p(x_1 \mid x_2) \sum_{x_5} P(x_5 \mid x_2) \sum_{x_4} P(x_4 \mid x_1) \sum_{x_3} P(x_3 \mid x_1) \\
&= 1
\end{aligned}$$

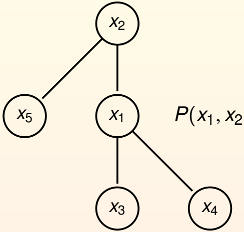
First Order Dependence Trees



Precedence Function

i	**j(i)**
1	2
5	2
3	1
4	1

First Order Dependence Tree



$$P(x_1, x_2, x_3, x_4, x_5) = p(x_1 \mid x_2)P(x_5 \mid x_2)P(x_3 \mid x_1)P(x_4 \mid x_1)P(x_2)$$

$$\begin{aligned}
[N] &= \{1, \ldots, N\} \\
M &\subset [N] \qquad j : M \to N \\
G &= ([N], E) \\
E &= \{\{j(m), m\} \mid m \in M\} \\
P(x_1, \ldots, x_N) &= P(x_m : m \in [N] - M) \prod_{m \in M} P(x_m \mid x_{j(m)})
\end{aligned}$$

Conditional Independence Assumption

Under the Markov assumption

$$
\begin{aligned}
P(x_i, x_{i+1}, \mid x_{i+2} \dots, x_N) &= \frac{P(x_i, \dots x_N)}{P(x_{i+2} \dots x_N)} \\
&= \frac{P(x_i \mid x_{i+1} \dots x_N) P(x_{i+1} \dots x_N)}{P(x_{i+2} \dots x_N)} \\
&= \frac{P(x_i \mid x_{i+1}) P(x_{i+1} \dots x_N)}{P(x_{i+2} \dots x_N)} \\
&= \frac{P(x_i \mid x_{i+1}) P(x_{i+1} \mid x_{i+2}) P(x_{i+2} \dots x_N)}{P(x_{i+2} \dots x_N)} \\
&= P(x_i \mid x_{i+1}) P(x_{i+1} \mid x_{i+2})
\end{aligned}
$$

Conditional Probability Products

$$P(x_1 \mid x_2, x_3, x_4)P(x_4 \mid x_2, x_5, x_6)P(x_5 \mid x_6)P(x_2, x_6 \mid x_3)P(x_3)$$

- Does this product make a probability function?
- If it does, is the probability function an extension of these conditional probabilities?

Summing

Define,

$$\begin{aligned}
Q(x_1,\ldots,x_6) &= P(x_1 \mid x_2,x_3,x_4)P(x_4 \mid x_2,x_5,x_6)P(x_5 \mid x_6) \\
&\quad P(x_2,x_6 \mid x_3)P(x_3)
\end{aligned}$$

$$\begin{aligned}
Q(x_2,\ldots,x_6) &= \sum_{x_1} Q(x_1,\ldots,x_6) \\
&= \sum_{x_1} P(x_1 \mid x_2,x_3,x_4)P(x_4 \mid x_2,x_5,x_6) \\
&\qquad P(x_5 \mid x_6)P(x_2,x_6 \mid x_3)P(x_3) \\
&= P(x_4 \mid x_2,x_5,x_6)P(x_5 \mid x_6)P(x_2,x_6 \mid x_3)P(x_3)
\end{aligned}$$

Summing

$$
\begin{aligned}
Q(x_2, x_3, x_5, x_6) &= \sum_{x_4} Q(x_2, \ldots, x_6) \\
&= \sum_{x_4} P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) P(x_2, x_6 \mid x_3) P(x_3) \\
&= P(x_5 \mid x_6) P(x_2, x_6 \mid x_3) P(x_3)
\end{aligned}
$$

Summing

$$
\begin{aligned}
Q(x_2, x_3, x_6) &= = \sum_{x_5} Q(x_2, x_3, x_5, x_6) \\
&= \sum_{x_5} P(x_5 \mid x_6) P(x_2, x_6 \mid x_3) P(x_3) \\
&= P(x_2, x_6 \mid x_3) P(x_3) \\
&= P(x_2, x_3, x_6) \\
\sum_{x_2, x_3, x_6} Q(x_2, x_3, x_6) &= \sum_{x_2, x_3, x_6} P(x_2, x_3, x_6) \\
&= 1
\end{aligned}
$$

Weak Extension

Define,

$$
\begin{aligned}
Q(x_1, \ldots, x_6) &= P(x_1 \mid x_2, x_3, x_4) P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) \\
&\quad P(x_2, x_6 \mid x_3) P(x_3)
\end{aligned}
$$

Is,

$$
\begin{aligned}
Q(x_1 \mid x_2, x_3, x_4) &= P(x_1 \mid x_2, x_3, x_4) \\
Q(x_4 \mid x_2, x_5, x_6) &= P(x_4 \mid x_2, x_5, x_6) \\
Q(x_5 \mid x_6) &= P(x_5 \mid x_6) \\
Q(x_2, x_6 \mid x_3) &= P(x_2, x_6 \mid x_3) \\
Q(x_3) &= P(x_3)
\end{aligned}
$$

Strong Extension

Define,

$$\begin{aligned} Q(x_1, \ldots, x_6) &= P(x_1 \mid x_2, x_3, x_4) P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) \\ &\quad P(x_2, x_6 \mid x_3) P(x_3) \end{aligned}$$

Is,

$$\begin{aligned} Q(x_1, x_2, x_3, x_4) &= P(x_1, x_2, x_3, x_4) \\ Q(x_4, x_2, x_5, x_6) &= P(x_4, x_2, x_5, x_6) \\ Q(x_5, x_6) &= P(x_5, x_6) \\ Q(x_2, x_3, x_6) &= P(x_2, x_3, x_6) \\ Q(x_3) &= P(x_3) \end{aligned}$$

Extension

Define,

$$Q(x_1,\ldots,x_6) = P(x_1 \mid x_2,x_3,x_4)P(x_4 \mid x_2,x_5,x_6) \\ P(x_5 \mid x_6)P(x_2,x_6 \mid x_3)P(x_3)$$

Is

$$Q(x_1 \mid x_2,x_3,x_4) = \frac{Q(x_1,x_2,x_3,x_4)}{Q(x_2,x_3,x_4)}$$

Find expressions for $Q(x_1,x_2,x_3,x_4)$ and $Q(x_2,x_3,x_4)$

$$\begin{aligned} Q(x_1,x_2,x_3,x_4) &= \sum_{x_5}\sum_{x_6} Q(x_1,\ldots,x_6) \\ &= \sum_{x_5}\sum_{x_6} P(x_1 \mid x_2,x_3,x_4)P(x_4 \mid x_2,x_5,x_6) \\ &\qquad P(x_5 \mid x_6)P(x_2,x_6 \mid x_3)P(x_3) \\ &= P(x_1 \mid x_2,x_3,x_4)\sum_{x_5}\sum_{x_6} P(x_4 \mid x_2,x_5,x_6) \end{aligned}$$

Extension

$$
\begin{aligned}
Q(x_2, x_3, x_4) &= \sum_{x_1} Q(x_1, x_2, x_3, x_4) \\
&= \sum_{x_1} P(x_1 \mid x_2, x_3, x_4) \sum_{x_5} \sum_{x_6} P(x_4 \mid x_2, x_5, x_6) \\
&\qquad P(x_5 \mid x_6) P(x_2, x_6 \mid x_3) P(x_3) \\
&= \sum_{x_5} \sum_{x_6} P(x_4 \mid x_2, x_5, x_6) \\
&\qquad P(x_5 \mid x_6) P(x_2, x_6 \mid x_3) P(x_3)
\end{aligned}
$$

Weak Extension

$$Q(x_1 \mid x_2, x_3, x_4) \quad = \quad \frac{Q(x_1, x_2, x_3, x_4)}{Q(x_2, x_3, x_4)}$$

$$= \quad \frac{P(x_1 \mid x_2, x_3, x_4) \sum_{x_5} \sum_{x_6} P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) P(x_2, x_3, x_6)}{\sum_{x_5} \sum_{x_6} P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) P(x_2, x_3, x_6)}$$

$$= \quad P(x_1 \mid x_2, x_3, x_4)$$

So we have shown a weak extension for one conditional probability.

Conditional Independence

Definition

Random variables X and Y are conditionally independent given random variable Z if and only if for all values x, y, z in the domain of the respective variables X, Y, Z

$$P(X = x, Y = y|Z = z) = P(X = x|Z = z)P(Y = y|Z = z)$$

For the sake of compactness, we write

- $P(x, y|z)$ for $P(X = x, Y = y \mid Z = z)$

Conditional Independence Notation

If random variables X and Y are conditionally independent of random variable Z we write

- $X \perp\!\!\!\perp Y \mid Z$

Let $\{X_1, \ldots, X_N\}$ be a set of random variables.
If X_i is conditionally independent of X_j given X_k we write

- $i \perp\!\!\!\perp j \mid k$

Let $A, B, C \subset \{1, \ldots, N\}$ with

- $A \cap B = \emptyset$
- $A \cap C = \emptyset$
- $B \cap C = \emptyset$

If $\{X_i : i \in A\}$ is conditionally independent of $\{X_j : j \in B\}$ given $\{X_k : k \in C\}$, then we write

- $A \perp\!\!\!\perp B \mid C$

Conditional Independence Characterization Theorem

Theorem

$P(x, y|z) = P(x|z)P(y|z)$ *if and only if* $P(x|y, z) = P(x|z)$

Proof.

Suppose $P(x, y|z) = P(x|z)P(y|z)$. *Consider* $P(x|y, z)$

$$\begin{aligned} P(x|y,z) &= \frac{P(x,y,z)}{P(y,z)} = \frac{P(x,y|z)P(z)}{P(y,z)} \\ &= \frac{P(x|z)P(y|z)P(z)}{P(y,z)} = P(x|z) \end{aligned}$$

Suppose $P(x|y, z) = P(x|z)$. *Consider* $P(x, y|z)$.

$$\begin{aligned} P(x,y|z) &= \frac{P(x,y,z)}{P(z)} = \frac{P(x|y,z)P(y,z)}{P(z)} \\ &= \frac{P(x|z)P(y,z)}{P(z)} = P(x|z)P(y|z) \end{aligned}$$

$\square$

Conditional Independence

Can we see if x_5 is conditionally independent of x_2 given x_6. Is $P(x_5 \mid x_2, x_6) = P(x_5 \mid x_6)$?

$$\begin{aligned}
Q(x_2, x_4, x_5, x_6) &= \sum_{x_1} \sum_{x_3} Q(x_1, \ldots, x_6) \\
&= \sum_{x_1} \sum_{x_3} P(x_1 \mid x_2, x_3, x_4) P(x_4 \mid x_2, x_5, x_6) \\
&\qquad P(x_5 \mid x_6) P(x_2, x_6 \mid x_3) P(x_3) \\
&= P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) P(x_2, x_6)
\end{aligned}$$

Extension

$$
\begin{aligned}
Q(x_2, x_5, x_6) &= \sum_{x_4} Q(x_2, x_4, x_5, x_6) \\
&= \sum_{x_4} P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) P(x_2, x_6) \\
&= P(x_5 \mid x_6) P(x_2, x_6)
\end{aligned}
$$

Extension

$$
\begin{aligned}
Q(x_5, x_6) &= \sum_{x_1} \sum_{x_2} \sum_{x_3} \sum_{x_4} Q(x_1, \ldots, x_6) \\
&= \sum_{x_1} \sum_{x_2} \sum_{x_3} \sum_{x_4} P(x_1 \mid x_2, x_3, x_4) P(x_4 \mid x_2, x_5, x_6) \\
&\qquad P(x_5 \mid x_6) P(x_2, x_6 \mid x_3) P(x_3) \\
&= \sum_{x_2} \sum_{x_3} \sum_{x_4} P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) P(x_2, x_3, x_6) \\
&= \sum_{x_2} \sum_{x_3} P(x_5 \mid x_6) P(x_2, x_3, x_6) \\
&= P(x_5 \mid x_6) P(x_6) = P(x_5, x_6)
\end{aligned}
$$

Conditional Independences

$$
\begin{aligned}
Q(x_2, x_5, x_6) &= P(x_5 \mid x_6) P(x_2, x_6) \\
Q(x_2, x_6) &= \sum_{x_5} Q(x_2, x_5, x_6) \\
&= \sum_{x_5} P(x_5 \mid x_6) P(x_2, x_6) \\
&= P(x_2, x_6) \\
Q(x_5 \mid x_2, x_6) &= \frac{Q(x_2, x_5, x_6)}{Q(x_2, x_6)} \\
&= \frac{P(x_5 \mid x_6) P(x_2, x_6)}{P(x_2, x_6)} = P(x_5 \mid x_6)
\end{aligned}
$$

Conditional Independence

Now,

$$Q(x_5 \mid x_2, x_6) = P(x_5 \mid x_6)$$

But,

$$Q(x_5, x_6) = P(x_5, x_6)$$

Hence,

$$Q(x_5 \mid x_6) = P(x_5 \mid x_6)$$

Therefore,

$$Q(x_5 \mid x_2, x_6) = Q(x_5 \mid x_6)$$

$$x_5 \perp\!\!\!\perp x_2 \mid x_6$$

Conditional Independence

Suppose, $x_5 \perp\!\!\!\perp x_2 \mid x_6$

$$Q(x_5 \mid x_2, x_6) = Q(x_5 \mid x_6)$$

Then,

$$Q(x_5, x_2 \mid x_6) = Q(x_5 \mid x_6) Q(x_2 \mid x_6)$$

$$\begin{aligned}
Q(x_5, x_2 \mid x_6) &= \frac{Q(x_2, x_5, x_6)}{Q(x_6)} \\
&= \frac{Q(x_5 \mid x_2, x_6) Q(x_2, x_6)}{Q(x_6)} \\
&= \frac{Q(x_5 \mid x_6) Q(x_2, x_6)}{Q(x_6)} \\
&= Q(x_5 \mid x_6) Q(x_2 \mid x_6)
\end{aligned}$$

Additional Relationships You Work Out

$$
\begin{aligned}
Q(x_4 \mid x_2, x_5, x_6) &= \frac{Q(x_2, x_4, x_5, x_6)}{Q(x_2, x_5, x_6)} \\
&= \frac{P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) P(x_2, x_6)}{P(x_5 \mid x_6) P(x_2, x_6)} \\
&= P(x_4 \mid x_2, x_5, x_6)
\end{aligned}
$$

Additional Relationships You Work Out

$$
\begin{aligned}
Q(x_2, x_3, x_6) &= \sum_{x_1} \sum_{x_4} \sum_{x_5} Q(x_1, \ldots, x_6) \\
&= \sum_{x_1} \sum_{x_4} \sum_{x_5} P(x_1 \mid x_2, x_3, x_4) P(x_4 \mid x_2, x_5, x_6) \\
&\qquad P(x_5 \mid x_6) P(x_2, x_6 \mid x_3) P(x_3) \\
&= \sum_{x_4} \sum_{x_5} P(x_4 \mid x_2, x_5, x_6) P(x_5 \mid x_6) P(x_2, x_3, x_6) \\
&= \sum_{x_5} P(x_5 \mid x_6) P(x_2, x_3, x_6) \\
&= P(x_2, x_3, x_6)
\end{aligned}
$$

Semi-Graphoid

Definition

Let I be an index set containing the indexes of all the variables. Let $\mathcal{G}$ be a collection of triples each of whose components are subsets of the index set I. $\mathcal{G}$ is called a Semi-Graphoid if and only if

- Mutual Exclusivity: $(A, B, C) \in \mathcal{G}$ implies
  - $A \cap B = \emptyset$, $A \cap C = \emptyset$, $B \cap C = \emptyset$
- Symmetry: $(A, B, C) \in \mathcal{G}$ if and only if $(B, A, C) \in \mathcal{G}$
- Decomposition: $(A, B \cup D, C) \in \mathcal{G}$ implies $(A, B, C) \in \mathcal{G}$
- Weak Union: $(A, B \cup C, D) \in \mathcal{G}$ implies $(A, B, C \cup D) \in \mathcal{G}$
- Contraction: $(A, B, C \cup D) \in \mathcal{G}$ and $(A, C, D) \in \mathcal{G}$ imply $(A, B \cup C, D) \in \mathcal{G}$

Conditional Independence and Semi-Graphoids

Theorem

Let $\{X_1, \ldots, X_N\}$ be a set of random variables. Let

$$\mathcal{G} = \{(A, B, C) \in [N]^3 \quad | \quad \begin{aligned} & A \cap B = \emptyset \\ & A \cap C = \emptyset \\ & B \cap C = \emptyset \\ & A \perp\!\!\!\perp B \mid C \} \end{aligned}$$

Then $\mathcal{G}$ is a semi-graphoid.

Proof.

We need to prove Symmetry, Decomposition, Weak Union, and Contraction. We do so in the following propositions where X, Y and Z represent tuples of random variables. $\square$

Conditional Independence: Symmetry

Proposition

$X \perp\!\!\!\perp Y \mid Z$ *implies* $Y \perp\!\!\!\perp X \mid Z$

Proof.

$X \perp\!\!\!\perp Y \mid Z$ *implies* $P(xy \mid z) = P(x \mid z)P(y \mid z)$

$P(xy \mid z) = P(x \mid z)P(y \mid z)$ *implies* $P(xy \mid z) = P(y \mid z)P(x \mid z)$

$P(xy \mid z) = P(y \mid z)P(x \mid z)$ *implies* $Y \perp\!\!\!\perp X \mid Z$ $\square$

Conditional Independence: Decomposition

Proposition

$Y \perp\!\!\!\perp Z_1, Z_2 \mid X$ implies $Y \perp\!\!\!\perp Z_1 \mid X$ and $Y \perp\!\!\!\perp Z_2 \mid X$.

Proof.

$$
\begin{aligned}
P(y, z_1, z_2 \mid x) &= P(y \mid x) P(z_1, z_2 \mid x) \\
\sum_{z_2} P(y, z_1, z_2 \mid x) &= \sum_{z_2} P(y \mid x) P(z_1, z_2 \mid x) \\
P(y, z_1 \mid x) &= P(y \mid x) P(z_1 \mid x)
\end{aligned}
$$

Hence, $Y \perp\!\!\!\perp Z_1 \mid X$.
The proof for $Y \perp\!\!\!\perp Z_2 \mid X$ is similar with the roles of Z_1 and Z_2 interchanged. □

Conditional Independence: Weak Union

Proposition

$Y \perp\!\!\!\perp Z_1, Z_2 \mid X$ *implies* $Y \perp\!\!\!\perp Z_1 \mid X, Z_2$ *and* $Y \perp\!\!\!\perp Z_2 \mid X, Z_1$.

Proof.

Suppose $Y \perp\!\!\!\perp Z_1, Z_2 \mid X$. *Consider* $P(Y, Z_1 \mid X, Z_2)$.

$$
\begin{aligned}
P(y, z_1 \mid x, z_2) &= \frac{P(x, y, z_1, z_2)}{P(x, z_2)} = \frac{P(y, z_1, z_2 \mid x)P(x)}{P(x, z_2)} \\
&= \frac{P(y \mid x)P(z_1, z_2 \mid x)P(x)}{P(x, z_2)} = P(y \mid x)P(z_1 \mid x, z_2)
\end{aligned}
$$

But $Y \perp\!\!\!\perp Z_1, Z_2 \mid X$ *implies* $Y \perp\!\!\!\perp Z_2 \mid X$ *so that* $P(y, z_2 \mid x) = P(y \mid x)P(z_2 \mid x)$. *Hence,* $P(y \mid x) = P(y, z_2 \mid x)/P(z_2 \mid x)$. *Therefore,*

$$
\begin{aligned}
P(y, z_1 \mid x, z_2) &= \frac{P(y, z_2 \mid x)}{P(z_2 \mid x)} P(z_1 \mid x, z_2) \\
&= P(y \mid x, z_2)P(z_1 \mid x, z_2)
\end{aligned}
$$

Thus, $Y \perp\!\!\!\perp Z_1 \mid X, Z_2$. *Similarly,* $Y \perp\!\!\!\perp Z_2 \mid X, Z_1$. □

Conditional Independence: Contraction

Proposition

$X \perp\!\!\!\perp Y \mid Z_1 \cup Z_2$ *and* $X \perp\!\!\!\perp Z_1 \mid Z_2$ *imply* $X \perp\!\!\!\perp Y \cup Z_1 \mid Z_2$

Proof.

$X \perp\!\!\!\perp Y \mid Z_1 \cup Z_2$ *implies* $P(xy \mid z_1, z_2) = P(x \mid z_1, z_2)P(y \mid z_1, z_2)$
$X \perp\!\!\!\perp Z_1 \mid Z_2$ *implies* $P(xz_1 \mid z_2) = P(x \mid z_2)P(z_1 \mid z_2)$

$$\begin{aligned}
P(xyz_1 \mid z_2) &= \frac{P(xyz_1z_2)}{P(z_2)} = \frac{P(xy \mid z_1z_2)P(z_1z_2)}{P(z_2)} \\
&= P(x \mid z_1z_2)P(y \mid z_1z_2)\frac{P(z_1z_2)}{P(z_2)} \\
&= P(xz_1z_2)\frac{P(y \mid z_1z_2)}{P(z_2)} = P(xz_1 \mid z_2)P(y \mid z_1z_2) \\
&= P(x \mid z_2)P(z_1 \mid z_2)\frac{P(yz_1z_2)}{P(z_1z_2)} \\
&= P(x \mid z_2)\frac{P(z_1z_2)}{P(z_2)}\frac{P(yz_1z_2)}{P(z_1z_2)} = P(x \mid z_2)P(yz_1 \mid z_2)
\end{aligned}$$

Semi-Graphoid

Theorem

$\mathcal{G}$ is a Semi-Graphoid if and only if

- $A \perp\!\!\!\perp B \mid C$ *if and only if* $B \perp\!\!\!\perp A \mid C$
- $A \perp\!\!\!\perp B \cup C \mid D$ *if and only if* $A \perp\!\!\!\perp B \mid C \cup D$ *and* $A \perp\!\!\!\perp C \mid D$

Graphoid

Definition

Let I be an index set containing the indexes of all the variables. Let G be a collection of triples each of whose components are subsets of the index set I. We write $A \perp\!\!\!\perp B \mid C$ if and only if the triple $(A, B, C) \in G$.

G is called a Graphoid if and only if

- Mutual Exclusivity: $(A, B, C) \in G$ implies
  - $A \cap B = \emptyset$, $A \cap C = \emptyset$, $B \cap C = \emptyset$
- Symmetry: $(A, B, C) \in \mathcal{G}$ if and only if $(B, A, C) \in \mathcal{G}$
- Decomposition: $(A, B \cup D, C) \in \mathcal{G}$ implies $(A, B, C) \in \mathcal{G}$
- Weak Union: $(A, B \cup C, D) \in \mathcal{G}$ implies $(A, B, C \cup D) \in \mathcal{G}$
- Contraction: $(A, B, C \cup D) \in \mathcal{G}$ and $(A, C, D) \in \mathcal{G}$ imply $(A, B \cup C, D) \in \mathcal{G}$
- Intersection: $(A, B, C \cup D) \in \mathcal{G}$ and $(A, C, B \cup D) \in \mathcal{G}$ imply $(A, B \cup C, D) \in \mathcal{G}$

Conditional Independence

Proposition

$Y \perp\!\!\!\perp Z_2 \mid X, Z_1$ and $Y \perp\!\!\!\perp Z_1 \mid X, Z_2$ imply
$P(y \mid x, z_2) = P(y \mid x, z_1)$ for all values x, y, z_1, z_2.

Proof.

By the Conditional Independence Characterization Theorem, $Y \perp\!\!\!\perp Z_2 \mid X, Z_1$ implies $P(y \mid x, z_1, z_2) = P(y \mid x, z_1)$ and with the roles of Z_1 and Z_2 interchanged, $Y \perp\!\!\!\perp Z_1 \mid X, Z_2$ implies $P(y \mid x, z_1, z_2) = P(y \mid x, z_2)$. Now, $P(y \mid x, z_1, z_2) = P(y \mid x, z_1)$ and $P(y \mid x, z_1, z_2) = P(y \mid x, z_2)$ imply $P(y \mid x, z_1) = P(y \mid x, z_2)$. $\square$

Conditional Independence

Proposition

If $P(y \mid x, z_1) = P(y \mid x, z_2)$ *for all values* x, y, z_1, z_2 *of the random variables* X, Y, Z_1, Z_2*, and* $P(x, y, z_1) > 0$*, and* $P(x, y, z_2) > 0$*,then* $P(y \mid x) = P(y \mid x, z_1) = P(y \mid x, z_2)$.

Proof.

$$\begin{aligned}
P(y \mid x, z_1) &= P(y \mid x, z_2) = \frac{P(x, y, z_2)}{P(x, z_2)} \\
P(y \mid x, z_1)P(x, z_2) &= P(x, y, z_2) \\
\sum_{z_2} P(y \mid x, z_1)P(x, z_2) &= \sum_{z_2} P(x, y, z_2) = P(x, y) \\
P(y \mid x, z_1)P(x) &= P(x, y) \\
P(y \mid x, z_1) &= P(y \mid x)
\end{aligned}$$

□

Conditional Independence: Intersection

Proposition

Suppose that for any values for any group of joint variables, their probability is greater than zero. Then, $Y \perp\!\!\!\perp Z_1 \mid X, Z_2$ and $Y \perp\!\!\!\perp Z_2 \mid X, Z_1$ imply $Y \perp\!\!\!\perp Z_1, Z_2 \mid X$. (The Intersection Property holds.)

Proof.

$$
\begin{aligned}
P(y, z_1, z_2 \mid x) &= \frac{P(y, z_1, z_2, x)}{P(x)} = \frac{P(y, z_1 \mid x, z_2)P(x, z_2)}{P(x)} \\
&= P(y \mid x, z_2)P(z_1 \mid x, z_2)\frac{P(x, z_2)}{P(x)} \\
&= P(y \mid x, z_2)P(z_1, z_2 \mid x)
\end{aligned}
$$

But by the previous corollary, $Y \perp\!\!\!\perp Z_1 \mid X, Z_2$ and $Y \perp\!\!\!\perp Z_2 \mid X, Z_1$ implies $P(y \mid x, z_2) = P(y \mid x)$. Hence,

$$
P(y, z_1, z_2 \mid x) = P(y \mid x)P(z_1, z_2 \mid x)
$$

□

Block Independence Theorem

Theorem

Suppose that for any values for any group of joint variables, the probability is greater than zero. Then, $Y \perp\!\!\!\perp Z_1 \cup Z_2 \mid X$ *if and only if* $Y \perp\!\!\!\perp Z_1 \mid X \cup Z_2$ *and* $Y \perp\!\!\!\perp Z_2 \mid X \cup Z_1$*.*

Proof.

By weak union, $Y \perp\!\!\!\perp Z_1 \cup Z_2 \mid X$ *implies* $Y \perp\!\!\!\perp Z_1 \mid X \cup Z_2$ *and* $Y \perp\!\!\!\perp Z_2 \mid Z_1 \cup X$*. By intersection,* $Y \perp\!\!\!\perp Z_1 \mid X \cup Z_2$ *and* $Y \perp\!\!\!\perp Z_2 \mid Z_1 \cup X$ *implies* $Y \perp\!\!\!\perp Z_1 \cup Z_2 \mid X$*.* □

Graphical Models

Graphical Models associates a graph, called the conditional independence graph, from which the all the conditional independencies can be easily seen.

When the conditional independence graph is triangulated, then the joint probability function can be expressed with a probability product form.

- The product form can be read off the graph
- The product form is a strong extension of the marginal terms of the product

Graphs

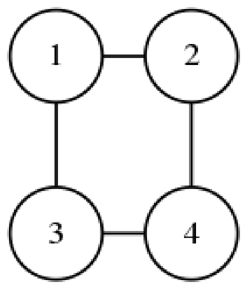
Definition

A *graph* $G = (N, E)$ where N is an index set and E, the edge set, is a collection of subsets of N where each subset has exactly 2 elements of N.

Graphs

Here, $G = (N, E)$ where

$$
\begin{aligned}
N &= \{1, 2, 3, 4\} \\
E &= \{\{1, 2\}, \{2, 4\}, \{3, 4\}, \{3, 1\}\}
\end{aligned}
$$

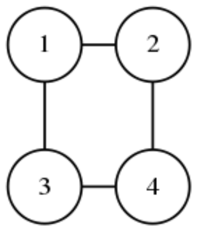


Boundary

Definition

Let $G = (N, E)$ be a graph and $i \in N$. The boundary of i is defined by

$$bndry(i) = \{j \in N \mid \{i, j\} \in E\}$$



- $bndry(1) = \{2, 3\}$
- $bndry(2) = \{1, 4\}$
- $bndry(3) = \{1, 4\}$
- $bndry(4) = \{2, 3\}$

Conditional Independence Graph: Definition

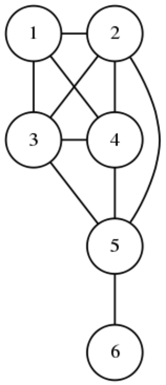
Definition

A graph (N, E) is called a Conditional Independence Graph of a random variable set $\mathcal{X} = \{X_1, \ldots, X_M\}$ if and only if $N = \{1, \ldots, M\}$, the index set for the variables in $\mathcal{X}$, and

$$E^c = \{\{i, j\} \mid X_i \perp\!\!\!\perp X_j \mid \mathcal{X} - \{X_i, X_j\}\}$$

Conditional Independence Graph

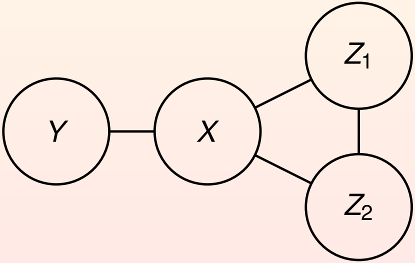
Nodes correspond to indexes of variables in the variable set
$\mathcal{X} = \{X_1, \ldots, X_6\}$
$\{i, j\}$ not in the edge set means $X_i \perp\!\!\!\perp X_j \mid \mathcal{X} - \{X_i, X_j\}$



Conditional Independence Graph

$\{Y, Z_1\}$ and $\{Y, Z_2\}$ not in edge set means

$$
\begin{aligned}
Y \perp\!\!\!\perp Z_1 &\mid \{X, Y, Z_1, Z_2\} - \{Y, Z_1\} \\
Y \perp\!\!\!\perp Z_2 &\mid \{X, Y, Z_1, Z_2\} - \{Y, Z_2\} \\
Y \perp\!\!\!\perp Z_1 &\mid \{X, Z_2\} \\
Y \perp\!\!\!\perp Z_2 &\mid \{X, Z_1\}
\end{aligned}
$$

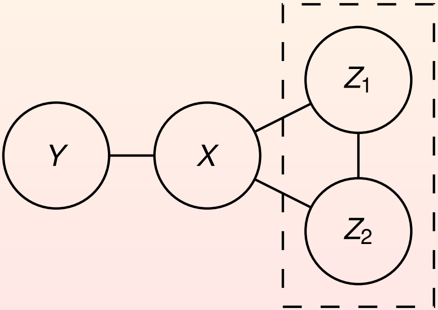


Block Independence Theorem

Y is conditionally independent of the block $\{Z_1, Z_2\}$ given X

Theorem

Suppose that for any values for any group of joint variables, the joint probability is greater than zero. $Y \perp\!\!\!\perp Z_1, Z_2 \mid X$ *if and only if* $Y \perp\!\!\!\perp Z_1 \mid X, Z_2$ *and* $Y \perp\!\!\!\perp Z_2 \mid X, Z_1$.

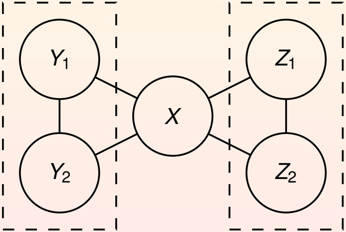


Reduction Theorem

Theorem

Suppose that for any values for any group of joint variables, the joint probability is greater than zero.

- $Y \perp\!\!\!\perp Z_1, Z_2 \mid X$ *if and only if* $Y \perp\!\!\!\perp Z_1 \mid X, Z_2$ *and* $Y \perp\!\!\!\perp Z_2 \mid X, Z_1$.
- $Y \perp\!\!\!\perp Z_1, Z_2 \mid X$ *implies* $Y \perp\!\!\!\perp Z_1 \mid X$ *and* $Y \perp\!\!\!\perp Z_2 \mid X$.



- $Y_1 \perp\!\!\!\perp Z_1 \mid X,\ Y_1 \perp\!\!\!\perp Z_2 \mid X,\ Y_2 \perp\!\!\!\perp Z_1 \mid X,\ Y_2 \perp\!\!\!\perp Z_2 \mid X$
- $Y_1, Y_2 \perp\!\!\!\perp Z_1 \mid X,\ Y_1, Y_2 \perp\!\!\!\perp Z_2 \mid X,\ Y_1, Y_2 \perp\!\!\!\perp Z_1, Z_2 \mid X$
- $Z_1, Z_2 \perp\!\!\!\perp Y_1 \mid X,\ Z_1, Z_2 \perp\!\!\!\perp Y_2 \mid X$

Paths

Definition

Let (G, E) be a graph and $g_1, \ldots, g_N \in G$. $< g_1, \ldots, g_N >$ is a path in (G, E) if and only if $\{g_n, g_{n+1}\} \in E$ for every $n \in \{1, \ldots, N-1\}$.

Connectedness

Definition

Let (G, E) be a graph and A, B be subsets of G. A and B are said to be connected if and only if for some $a \in A$ and $b \in B$, there is a path $< a, g_1, \ldots, g_N, b >$ in G.

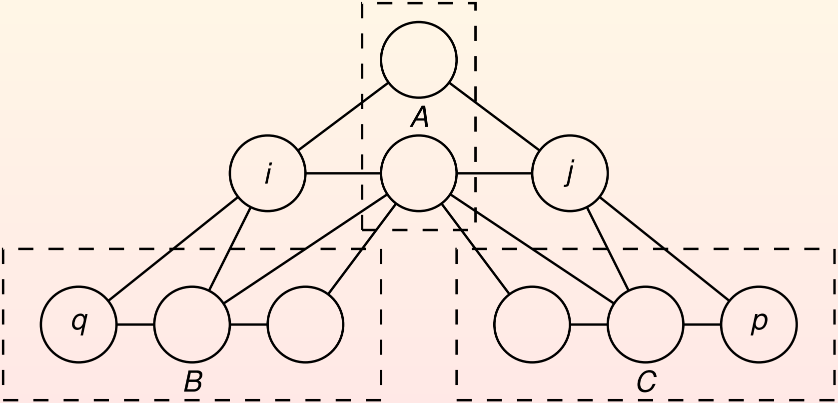
Separation

Definition

Let (G, E) be a graph and A, B, S be non-empty subsets of G. S separates A from B if and only if for every $a \in A$ and $b \in B$, every path in G that begins with a and ends with b has at least one node in S.

Separation Theorem

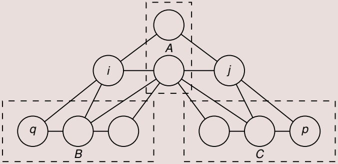
A separates $B \cup \{i\}$ from $C \cup \{j\}$
$N = A \cup B \cup C \cup \{i, j\}$
Then $i \perp\!\!\!\perp j \mid A$



Separation Theorem

Theorem

Let $G = (N, E)$ be a connected conditional independence graph for a set of random variables whose joint probability is positive. If $A \subset N$ is any node set that separates two nodes i and j, then $i \perp\!\!\!\perp j \mid A$.



Proof.

Let B be the set of nodes that either connect to i directly or through A. Let C be the set of nodes that either connect to j directly or through A. Hence, $\{A, B, C, \{i, j\}\}$ form a partition of N. By construction of the conditional independence graph, $i \perp\!\!\!\perp j \mid N - \{i, j\}$ and $i \perp\!\!\!\perp p \mid N - \{i, p\}$. Application of the block independence theorem yields $i \perp\!\!\!\perp j, p \mid N - \{i, j, p\}$. Application of the reduction theorem yields $i \perp\!\!\!\perp j \mid N - \{i, j, p\}$. Repeated application using the remaining nodes of C yields $i \perp\!\!\!\perp j \mid N - \{i, j\} - C$. Similarly for using q. Repeated application yields $i \perp\!\!\!\perp j \mid N - \{i, j\} - B - C$. But $N - \{i, j\} - B - C = A$. Therefore $i \perp\!\!\!\perp j \mid A$. $\square$

Local Markov Property

All conditional independences can be read off the Conditional Independence Graph.

Corollary

Let $G = (N, E)$ be a conditional independence graph and $n \in N$. Define $B = N - \{n\} - bndry(n)$. Then $n \perp\!\!\!\perp B \mid bndry(n)$.

Proof.

The set $bndry(n)$ separates n from B. □

Definition

Let $G = (N, E)$ be a conditional independence graph and $n \in N$. The Markov Blanket of node n is $bndry(n)$.

Complete Graphs

Definition

A graph $G = (N, E)$ is complete if and only if

$$E = \{\{i,j\} \mid i,j \in N, i \neq j\}$$

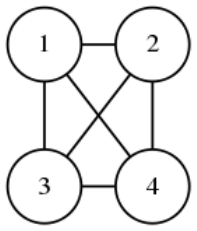


Figure: The Complete Graph on 4 Nodes

Graph Restriction

Definition

Let $G = (N, E)$ be a graph and $A \subset N$. The graph of G **restricted** to A, $G\mid_A$, is defined by

$$G\mid_A = (A, E\mid_A)$$

where

$$E\mid_A = \{\{i, j\} \in E \mid i, j \in A\}$$

Completeness

Definition

Let $G = (N, E)$ be a graph. Let a subset of nodes $A \subset N$ be given. We say A is complete if and only if $G|_A$ is a complete graph.

Maximally Complete

Definition

A subset of nodes $A \subset N$ is maximally complete if and only if

- $G\,|_A$ is complete
- $B \supset A$ and $G\,|_B$ complete implies $B = A$

Clique

Definition

Let $G = (N, E)$ be a graph. A maximally complete subset $A \subset N$ is called a clique of G.

Chordal Graphs

Definition

A graph is chordal (triangulated, decomposable) if and only if every cycle of length 4 or more has a chord.

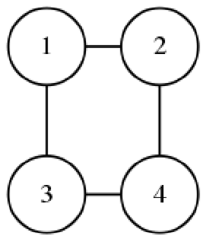


Figure: Non-chordal

Chordal Graphs

Definition

A graph is chordal (triangulated, decomposable) if and only if every cycle of length 4 or more has a chord.

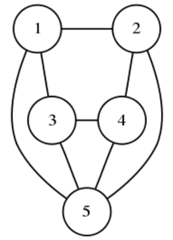


Figure: Non-chordal

Decomposable Graphs

Definition

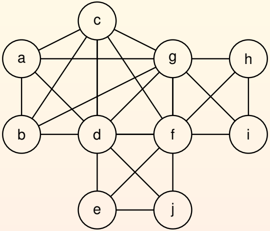
A Graph $G = (N, E)$ is Decomposable if and only if

- G is chordal
- The cliques of G can be put in running intersection order $C_1, \ldots, C_K$ with separators $S_2, \ldots S_K$ where

$$S_k = C_k \bigcap \left(\bigcup_{i=1}^{k-1} C_i \right), k = 2, \ldots, K-1$$

  such that S_k is complete.

Example

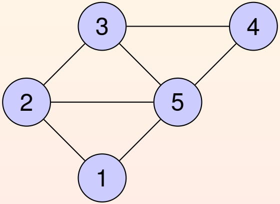


C_1	$=$	$\{a, b, c, d, g\}$					
C_2	$=$	$\{c, d, f, g\}$	S_2	$=$	$C_2 \cap C_1$	$=$	$\{c, d, g\}$
C_3	$=$	$\{f, g, h, i\}$	S_3	$=$	$C_3 \cap (C_1 \cup C_2)$	$=$	$\{f, g\}$
C_4	$=$	$\{d, e, f, j\}$	S_4	$=$	$C_4 \cap (C_1 \cup C_2 \cup C_3)$	$=$	$\{d, f\}$

Decomposable Graph

$$I = \{1, 2, 3, 4, 5\}$$

C_1	$=$	$\{1, 2, 5\}$	$1 \perp\!\!\!\perp 4$	$\mid$	$2, 5$
C_2	$=$	$\{2, 3, 5\}$	$1 \perp\!\!\!\perp 3$	$\mid$	$2, 5$
C_3	$=$	$\{3, 4, 5\}$	$2 \perp\!\!\!\perp 4$	$\mid$	$3, 5$
S_2	$=$	$\{2, 5\}$	$1 \perp\!\!\!\perp 4$	$\mid$	$3, 5$
S_3	$=$	$\{3, 5\}$	$1 \perp\!\!\!\perp 4$	$\mid$	$2, 3, 5$



$$\begin{aligned} P(x_i : i \in I) &= \frac{P(x_i : i \in C_1)P(x_i : i \in C_2)P(x_i : i \in C_3)}{P(x_i : i \in S_2)P(x_i : i \in S_3)} \\ &= P(x_i : i \in C_1)P(x_i : i \in C_2 - S_2 \mid S_2)P(x_i : i \in C_3 - S_3 \mid S_3) \end{aligned}$$

Notation

Let I be an index subset. If $I = \{1, 3, 7\}$, then

$$P(x_i : i \in I) = P(x_1, x_3, x_7)$$

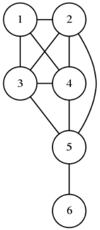
Decomposable Graphs

Theorem

If G is a decomposable graph with cliques in running intersection order $C_1, \dots, C_K$ and separators $S_2, \dots, S_K$ then

$$
\begin{aligned}
P(x_1, \dots, x_N) &= \frac{\prod_{k=1}^{K} P(x_i : i \in C_k)}{\prod_{m=2}^{K} P(x_j : j \in S_m)} \\
&= P(x_i : i \in C_1) \prod_{k=2}^{K} P(x_i : i \in C_k - S_k \mid S_k)
\end{aligned}
$$

Example



Cliques in running intersection order: $\{1, 2, 3, 4\}, \{2, 3, 4, 5\}, \{5, 6\}$
Separators: $\{2, 3, 4\}, \quad \{5\}$

$$P(x_1, \ldots, x_6) = P(x_1, x_2, x_3, x_4)P(x_5 \mid x_2, x_3, x_4)P(x_6 \mid x_5)$$

Product Form

The product form

$$Q(x_1, \dots, x_6) = P(x_1, x_2, x_3, x_4) P(x_5 \mid x_2, x_3, x_4) P(x_6 \mid x_5)$$

is an extension of the marginals

- $P(x_1, x_2, x_3, x_4)$
- $P(x_2, x_3, x_4, x_5)$
- $P(x_5, x_6)$

Product Form

$$Q(x_1,\ldots,x_6) = P(x_1,x_2,x_3,x_4)P(x_5 \mid x_2,x_3,x_4)P(x_6 \mid x_5)$$

$$\begin{aligned}
Q(x_1,x_2,x_3,x_4) &= \sum_{x_5}\sum_{x_6} Q(x_1,\ldots,x_6) \\
&= \sum_{x_5}\sum_{x_6} P(x_1,x_2,x_3,x_4)P(x_5 \mid x_2,x_3,x_4)P(x_6 \mid x_5) \\
&= P(x_1,x_2,x_3,x_4)\sum_{x_5} P(x_5 \mid x_2,x_3,x_4)\sum_{x_6} P(x_6 \mid x_5) \\
&= P(x_1,x_2,x_3,x_4)\sum_{x_5} P(x_5 \mid x_2,x_3,x_4) \\
&= P(x_1,x_2,x_3,x_4)
\end{aligned}$$

Product Form

$$Q(x_1, \ldots, x_6) = P(x_1, x_2, x_3, x_4)P(x_5 \mid x_2, x_3, x_4)P(x_6 \mid x_5)$$

$$\begin{aligned}
Q(x_2, x_3, x_4, x_5) &= \sum_{x_1} \sum_{x_6} P(x_1, x_2, x_3, x_4)P(x_5 \mid x_2, x_3, x_4)P(x_6 \mid x_5) \\
&= P(x_5 \mid x_2, x_3, x_4) \sum_{x_1} P(x_1, x_2, x_3, x_4) \sum_{x_6} P(x_6 \mid x_5) \\
&= P(x_5 \mid x_2, x_3, x_4)P(x_2, x_3, x_4) = P(x_2, x_3, x_4, x_5)
\end{aligned}$$

Product Form

$$Q(x_1, \ldots, x_6) = P(x_1, x_2, x_3, x_4)P(x_5 \mid x_2, x_3, x_4)P(x_6 \mid x_5)$$

$$\begin{aligned}
Q(x_2, x_3, x_4, x_5, x_6) &= \sum_{x_1} Q(x_1, \ldots, x_6) \\
&= \sum_{x_1} P(x_1, x_2, x_3, x_4)P(x_5 \mid x_2, x_3, x_4)P(x_6 \mid x_5) \\
&= P(x_2, x_3, x_4)P(x_5 \mid x_2, x_3, x_4)P(x_6 \mid x_5) \\
&= P(x_2, x_3, x_4, x_5)P(x_6 \mid x_5) \\
Q(x_5, x_6) &= \sum_{x_2} \sum_{x_3} \sum_{x_4} P(x_2, x_3, x_4, x_5)P(x_6 \mid x_5) \\
&= P(x_5)P(x_6 \mid x_5) = P(x_5, x_6)
\end{aligned}$$

Decomposable Graphs

$$
\begin{aligned}
S_k &= C_k \bigcap (\bigcup_{i=1}^{k-1} C_i), k = 2, \ldots, K \\
P(x_1, \ldots, x_N) &= P(x_i : i \in C_1) \prod_{k=2}^{K} P(x_i : i \in C_k - S_k \mid S_k)
\end{aligned}
$$

Proposition

$(C_k - S_k) \cap (\bigcup_{i=1}^{k-1} C_i) = \emptyset$

Proof.

$$
\begin{aligned}
(C_k - S_k) \cap (\cup_{i=1}^{k-1} C_i) &= (C_k - (C_k \cap (\cup_{i=1}^{k-1} C_i)) \cap (\cup_{i=1}^{k-1} C_i) \\
&= (C_k - (\cup_{i=1}^{k-1} C_i)) \cap (\cup_{i=1}^{k-1} C_i) \\
&= \emptyset
\end{aligned}
$$

Decomposable Graphs: Summability

$$
\begin{aligned}
S_k &= C_k \cap (\cup_{i=1}^{k-1} C_i), k = 2, \ldots, K \\
P(x_1, \ldots, x_N) &= P(x_i : i \in C_1) \prod_{k=2}^{K} P(x_i : i \in C_k - S_k \mid S_k) \\
(C_k - S_k) \cap (\cup_{i=1}^{k-1} C_i) &= \emptyset
\end{aligned}
$$

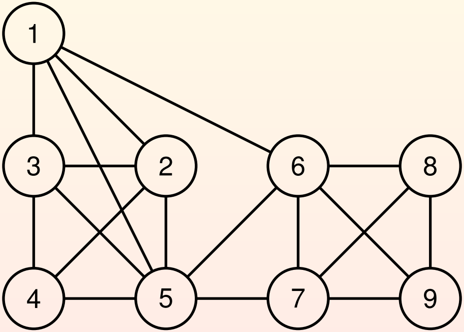
Proposition

$\sum_{x_1} \sum_{x_2} \cdots \sum_{x_N} P(x_i : i \in C_1) \prod_{k=2}^{K} P(x_i : i \in C_k - S_k \mid S_k) = 1$

Proof.

$$
\begin{aligned}
S &= \sum_{x_1} \sum_{x_2} \cdots \sum_{x_N} P(x_i : i \in C_1) \prod_{k=2}^{K} P(x_i : i \in C_k - S_k \mid S_k) \\
&= \sum_{C_1} \sum_{C_2 - S_2} \cdots \sum_{C_K - S_K} P(x_i : i \in C_1) \prod_{k=2}^{K} P(x_i : i \in C_k - S_k \mid S_k) \\
&= \sum_{C_1} P(x_i : i \in C_1) \sum_{C_2 - S_2} P(x_i : i \in C_2 - S_2 \mid S_2) \cdots \sum_{C_K - S_K} P(x_i : i \in C_K - S_K \mid S_K) \\
&= 1
\end{aligned}
$$

Summability Example



$$
\begin{aligned}
C_1 &= \{1,2,3,5\} & & \\
C_2 &= \{2,3,4,5\} & S_2 &= \{2,3,5\} \\
C_3 &= \{1,5,6\} & S_3 &= \{1,5\} \\
C_4 &= \{5,6,7\} & S_4 &= \{5,6\} \\
C_5 &= \{6,7,8,9\} & S_5 &= \{6,7\}
\end{aligned}
$$

$$
\begin{aligned}
S &= \textstyle\sum_{x_1} \cdots \sum_{x_9} P(x_1 x_2 x_3 x_5) P(x_4 | x_2 x_3 x_5) P(x_6 | x_1 x_5) P(x_7 | x_5 x_6) P(x_9 | x_6 x_7) \\
&= \textstyle\sum_{x_1 x_2 x_3 x_5} P(x_1 x_2 x_3 x_5) \sum_{x_4} P(x_4 | x_2 x_3 x_5) \sum_{x_6} P(x_6 | x_1 x_5) \sum_{x_7} P(x_7 | x_5 x_6) \sum_{x_8 x_9} P(x_8 x_9 | x_6 x_7) \\
&= 1
\end{aligned}
$$

Separators

Definition

Let $G = (V, E)$ be a connected graph. A non-empty subset $S \subset V$ is called a Separator of G if and only if $G(V - S, E|_{V-S})$ is not connected. Let A, B, and S be disjoint non-empty subsets of V. S is a Separator of A from B in graph G if and only if in the restricted graph $G|_{V-S}$, there exists no $a \in A$ and $b \in B$ such that $\{a, b\} \in E|_{V-s}$.
A separator S is called a Minimal Separator if and only if T a separator with $T \subset S$ implies $T = S$.

Theorem

A graph is triangulated if and only if each minimal separator is maximally complete.

Triangulated Graphs

Theorem

G is a triangulated graph if and only if the vertices of G can be partitioned into three nonempty subsets A, S, and B, such that

- *$G|_{A \cup S}$ and $G|_{B \cup S}$ are triangulated subgraphs of G*
- *S separates A from B*

This is one of the reasons that triangulated graphs are called decomposable graphs.

Triangulated Graphs

Definition

Let $G(V, E)$ be a graph and $\{A, B, S\}$ be a non-trivial partition of V. (A, B, S) is called a Decomposition of G into $G_{A \cup S}$ and $G_{B \cup S}$ if and only if

- S separates A from B in G
- G_S is a complete graph
- $G_{A \cup S}$ and $G_{B \cup S}$ are each triangulated

Decomposable Graphs

Theorem

A graph is decomposable if and only if either G is complete or there exists a decomposition (A, B, S) of G into $G_{A \cup S}$ and $G_{B \cup S}$.

Triangulated Graphs

Definition

A Perfect Elimination Ordering in a graph is an ordering of the vertices of the graph such that, for each vertex v, v and the neighbors of v that occur after v in the ordering form a maximally complete graph.

Theorem

A graph is triangulated if and only if it has a perfect elimination ordering.

Theorem

A graph is triangulated if and only if its cliques can be put in running intersection order.

Triangulated Graphs and Clique Finding

A triangulated graph can have only linearly many cliques, while non-chordal graphs may have exponentially many. Therefore clique finding in triangulated graphs can be done in polynomial time.

Triangulated Graphs

Theorem

If a graph G is triangulated graph and $C_1, \dots, C_K$ are the cliques of G put in running intersection order with separators $S_2, \dots, S_K$,

$$S_k = C_k \bigcap \left(\bigcup_{i=1}^{k-1} C_i \right), k = 2, \dots, K$$

then

$$P(x_1, \dots, x_N) = \frac{\prod_{k=1}^{K} P(x_i : i \in C_k)}{\prod_{k=2}^{K} P(x_i : i \in S_k)}$$

Conditional Independence Graphs

Theorem

Let $P(x_1, \ldots, x_N) > 0$ and G be the conditional independence graph of P. If $\{A, B, S\}$ is a non-trivial partition of $\{1, \ldots, N\}$ and S is a separator of A from B in G, then $A \perp\!\!\!\perp B \mid S$

$$P(x_i : i \in A \cup B | x_j : j \in S) = P(x_i : i \in A | x_j : j \in S) P(x_i : i \in B | x_j : j \in S)$$

Generalized Products

What happens if the conditional independence graph is not triangulated? Can the joint probability distribution be written in a product form?

Generalized Products

Theorem

Let f be a probability distribution. Then X is Conditionally Independent of Y given Z if and only if

$$f(x,y,z) = g(x,z)h(y,z)$$

Proof.

By definition of conditional independence, X is conditionally independent of Y given Z if and only if

$$f(x,y|z) = f(x|z)f(y|z)$$

Hence X is conditionally independent of Y given Z if and only if

$$\begin{aligned} f(x,y,z) &= f(x|z)f(y|z)f(z) \\ &= [f(x|z)][f(y|z)f(z)] \\ &= [f(x|z)][f(y,z)] \end{aligned}$$

Take $g(x,z) = f(x|z)$ and $h(y,z) = f(y,z)$ $\square$

Generalized Products

Definition

Let $B_1, \ldots, B_K$ be index subsets of {1,. . . ,N}. The product form $\prod_{k=1}^{K} a_k(x_i : i \in B_k)$ is called a *generalized product form* if and only if for some probability function $P(x_1, \ldots, x_N)$

- $P(x_1, \ldots, x_N) = \prod_{k=1}^{K} a_k(x_i : i \in B_k)$
- $P(x_1, \ldots, x_N)$ is an extension of $P(x_i : i \in B_k), k = 1, \ldots, K$

Generalized Products

Let $B_1, \ldots, B_K$ be index subsets of {1,...,N}. Given marginal probability functions $P(x_i : i \in B_k), k = 1, \ldots, K$ find functions $a_k(x_i : i \in B_k)$ such that

- $P(x_1, \ldots, x_N) = \prod_{k=1}^{K} a_k(x_i : i \in B_k)$
- $P(x_1, \ldots, x_N)$ is an extension of $P(x_i : i \in B_k), k = 1, \ldots, K$

Projections

Definition

Let $S = \{s_1, \ldots, s_M\}$ be an index subset of $\{1, \ldots, N\}$. $\pi_S(x_1, \ldots, x_N)$ is called the *projection* of $(x_1, \ldots, x_N)$ onto the index set S. $\pi_S(x) = (x_{s_1}, \ldots, x_{s_M}) = (x_i : i \in S)$.

If $(x_1, x_2, x_3, x_4, x_5) = (1, 5, 4, 3, 0)$ and $S = \{1, 4, 5\}$, then $\pi_S(1, 5, 4, 3, 0) = (x_i : i \in S) = (1, 3, 0)$.

Inverse Projection

Definition

Let h be a tuple whose components are indexed in index set S. Let I be the index set for all the variables. The *inverse projection* $\pi_I^{-1} h$ of h with respect to I is defined by

$$\pi_I^{-1}(h) = \{(x_1, \ldots, x_N) \mid \pi_S(x_1, \ldots, x_N) = h\}$$

Marginals

Let P be a probability function on N variables $(x_1, \ldots, x_N)$. Let $S_0, \ldots, S_{K-1}$ be K index sets of $\{1, \ldots, N\}$ covering $\{1, \ldots, N\}$. Fix k. Let h be a tuple whose components are indexed in index set S_k: $h = (x_i : i \in S_k)$.

$$P(h) = P(x_i : i \in S_k) = \sum_{(x_1, \ldots, x_N) \in \pi_I^{-1}(h)} P(x_1, \ldots, x_N)$$

Extension

Definition

Let P be a probability function on N variables $(x_1, \ldots, x_N)$. Let $S_0, \ldots, S_{K-1}$ be K index sets of $I = \{1, \ldots, N\}$. Let f_k be marginal probability functions defined on tuples $h_k = (x_i : i \in S_k)$, $k = 0, \ldots, K-1$. P is an *extension* of marginals $f_1, \ldots, f_K$ if and only if

$$f_k(h_k) = \sum_{(x_1,\ldots,x_N)\in\pi_I^{-1}(h_k)} P(x_1, \ldots, x_N)$$

Iterative Proportional Fitting

Let $(j) = j \mod K$. Let $S_0, \ldots, S_{K-1}$ be K index sets of $I = \{1, \ldots, N\}$. Let the range sets for the variables be $L_1, \ldots, L_N$. Let f_k be marginal probability functions defined on tuples $h_k = (x_i : i \in S_k) \in \bigtimes_{i \in S_k} L_i$, $k = 0, \ldots, K-1$. Let a_k be defined on the variables indexed by S_k, $a_k : \bigtimes_{i \in S_k} L_i \to [0,1]$ satisfy

$$\sum_{(x_1,\ldots,x_N)} \prod_{k=0}^{K-1} a_k(\pi_{S_k}(x_1, \ldots, x_N)) = 1$$

For $j \geq K-1$, iterative proportional fitting defines $a_K, a_{K+1}, \ldots$, $a_m : \bigtimes_{i \in S_{(m)}} L_i \to [0,1]$, $m = K, K+1, \ldots$, by

$$a_{j+1}(h) = \frac{f_{(j+1)}(h)}{\sum_{(x_1,\ldots,x_N) \in \pi_I^{-1}(h)} \prod_{m=j+2-K}^{j} a_m(\pi_{S_{(m)}}(x_1, \ldots, x_N))}$$

Iterative Proportional Fitting

For $j \geq K - 1$, iterative proportional fitting defines $a_K, a_{K+1}, \ldots,$ $a_m : \bigtimes_{i \in S_{(m)}} L_i \to [0,1]$, $m = K, K+1, \ldots$, by

$$a_{j+1}(h) = \frac{f_{(j+1)}(h)}{\sum_{(x_1,\ldots,x_N)\in\pi_I^{-1}(h)} \prod_{m=j+2-K}^{j} a_m(\pi_{S_{(m)}}(x_1,\ldots,x_N))}$$

$$\begin{array}{ccccccccccc} a_0 & a_1 & \ldots & a_{K-1} & a_K & a_{K+1} & \ldots & a_{2K-1} & a_{2K} & a_{2K+1} & \ldots \\ f_0 & f_1 & \ldots & f_{K-1} & f_0 & f_1 & \ldots & f_{K-1} & f_0 & f_1 & \ldots \end{array}$$

Iterative Proportional Fitting

For $j \geq K - 1$, iterative proportional fitting defines $a_K, a_{K+1}, \ldots,$ $a_m : \bigtimes_{i \in S_{(m)}} L_i \to [0,1]$, $m = K, K+1, \ldots$, by

$$a_{j+1}(h) = \frac{f_{(j+1)}(h)}{\sum_{(x_1,\ldots,x_N) \in \pi_I^{-1}(h)} \prod_{m=j+2-K}^{j} a_m(\pi_{S_{(m)}}(x_1, \ldots, x_N))}$$

$j + 2 - K \ldots, j$ indexes the last $K - 1$ a functions not including the a function associated with $f_{(j+1)}$

$$\hat{f}_{(j+1)}^{j+1}(h) = \sum_{(x_1,\ldots,x_N) \in \pi_I^{-1}(h)} \prod_{m=j+2-K}^{j+1} a_m(\pi_{S_{(m)}}(x_1, \ldots, x_N))$$

Example

Let x_1, x_2, x_3 be three binary $\{0, 1\}$ valued variables. Let marginals $f_0(x_1, x_2)$, $f_1(x_1, x_3)$, $f_2(x_2, x_3)$ be given. The a functions are defined on the same domains as the marginals.

$$a_0(x_1, x_2), a_1(x_1, x_3), a_2(x_2, x_3)$$

$$\begin{aligned}
(x_1, x_2) = (0, 0):\ a_3(0, 0) &= \frac{f_0(0, 0)}{a_1(0, 0)a_2(0, 0) + a_1(0, 1)a_2(0, 1)} \\
(x_1, x_2) = (0, 1):\ a_3(0, 1) &= \frac{f_0(0, 1)}{a_1(0, 0)a_2(1, 0) + a_1(0, 1)a_2(1, 1)}
\end{aligned}$$

Iterative Proportional Fitting

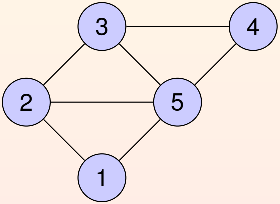
$$a_{j+1}(h) = \frac{f_{(j+1)}(h)}{\sum_{(x_1,\ldots,x_N)\in\pi^{-1}_{S_{(j+1)}}(h)} \prod_{m=j+2-K}^{j} a_m(x_i : i \in S_{(m)})}$$

- $P^{j+1}(x_1,\ldots,x_N) = \prod_{m=j+2-K}^{j+1} a_m(x_i : i \in S_{(m)})$ is a probability function and extension of $f_{(j+1)}$
- The iterative process converges
- In the limit, P^j is an extension of all the marginals $f_0,\ldots f_{K-1}$
- It is the unique minimal information extension

Decomposable Graph

$$I = \{1,2,3,4,5\}$$

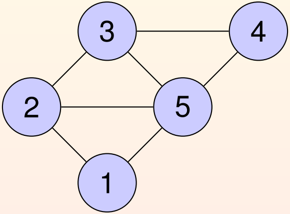
C_1	$=$	$\{1,2,5\}$	$1 \perp\!\!\!\perp 4$	$\mid$	$2,5$
C_2	$=$	$\{2,3,5\}$	$1 \perp\!\!\!\perp 3$	$\mid$	$2,5$
C_3	$=$	$\{3,4,5\}$	$2 \perp\!\!\!\perp 4$	$\mid$	$3,5$
S_2	$=$	$\{2,5\}$	$1 \perp\!\!\!\perp 4$	$\mid$	$3,5$
S_3	$=$	$\{3,5\}$	$1 \perp\!\!\!\perp 4$	$\mid$	$2,3,5$



$$\begin{aligned} P(x_i : i \in I) &= \frac{P(x_i : i \in C_1)P(x_i : i \in C_2)P(x_i : i \in C_3)}{P(x_i : i \in S_2)P(x_i : i \in S_3)} \\ &= P(x_i : i \in C_1)P(x_i : i \in C_2 - S_2 \mid S_2)P(x_i : i \in C_3 - S_3 \mid S_3) \end{aligned}$$

Decomposable Graph

In the conditional independence graph, there is an edge between node i and j if and only if X_i and X_j are conditionally independent given the rest of the variables.



$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{15}(x_1, x_5) P_{2|15}(x_2 \mid x_1, x_5) P_{3|25}(x_3 \mid x_2, x_5) P_{4|35}(x_4 \mid x_3, x_5) \end{aligned}$$

$$\{235 : 25\}, \{345 : 35\}$$

$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{15}(x_1, x_5) P_{2|15}(x_2 \mid x_1, x_5) P_{3|25}(x_3 \mid x_2, x_5) P_{4|35}(x_4 \mid x_3, x_5) \end{aligned}$$

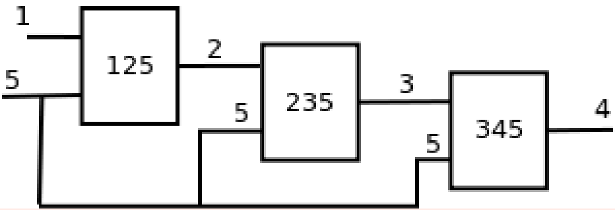


Figure: 1:System H

System Diagram 2

$$\{235 : 25\}, \{345 : 35\}$$

$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{25}(x_2, x_5) P_{1|25}(x_1 \mid x_2, x_5) P_{3|25}(x_3 \mid x_2, x_5) P_{4|35}(x_4 \mid x_3, x_5) \end{aligned}$$

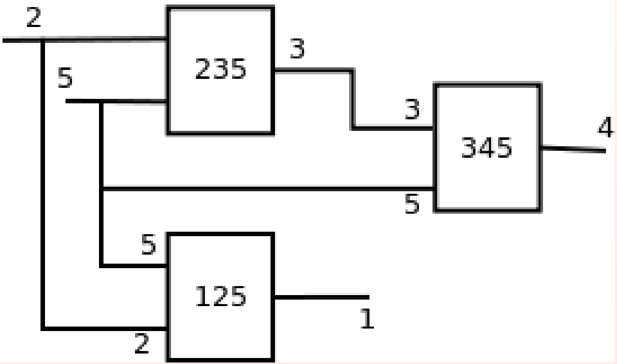


Figure: 1:System G

System Diagram 3

$$\{235 : 25\}, \{345 : 35\}$$

$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{12}(x_1, x_2) P_{5|12}(x_5 \mid x_1, x_2) P_{3|25}(x_3 \mid x_2, x_5) P_{4|35}(x_4 \mid x_3, x_5) \end{aligned}$$

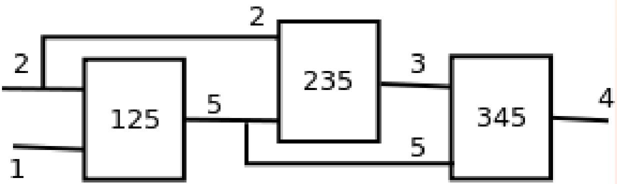


Figure: 1:System I

$$\{125 : 25\}, \{235 : 35\}$$

$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{1|25}(x_1 \mid x_2, x_5) P_{2|35}(x_2 \mid x_3, x_5) P_{4|35}(x_4 \mid x_3, x_5) P_{35}(x_3, x_5) \end{aligned}$$

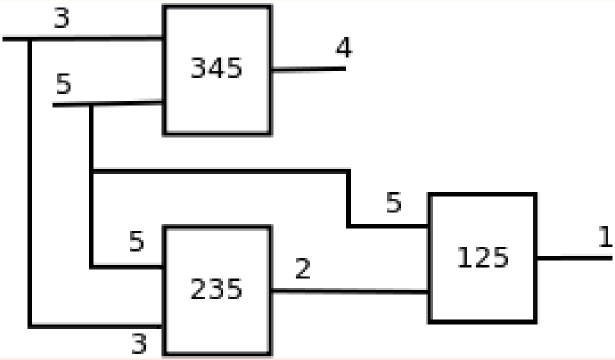


Figure: 2: System E

System Diagram 5

$$\{125 : 25\}, \{235 : 35\}$$

$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{1|25}(x_1 \mid x_2, x_5) P_{2|35}(x_2 \mid x_3, x_5) P_{3|45}(x_3 \mid x_4, x_5) P_{45}(x_4, x_5) \end{aligned}$$

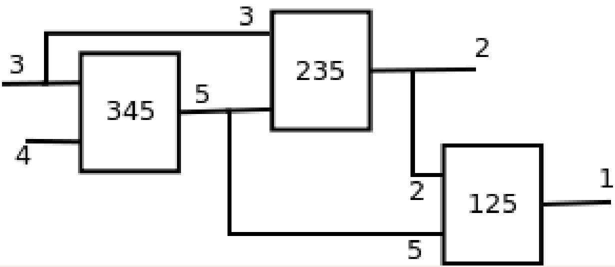
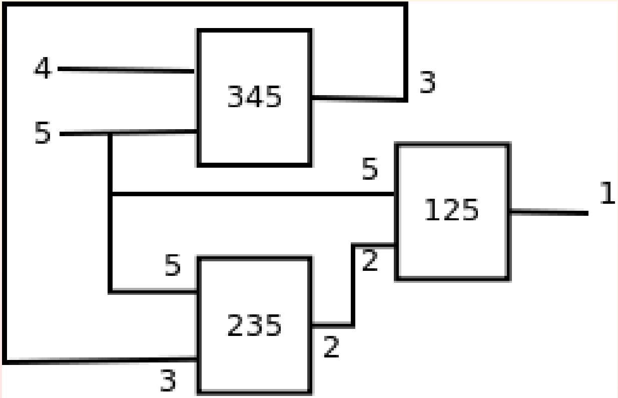


Figure: 2:System L

System Diagram 6

$$\{125 : 25\}, \{235 : 35\}$$

$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{1|25}(x_1 \mid x_2, x_5) P_{2|35}(x_2 \mid x_3, x_5) P_{5|34}(x_5 \mid x_3, x_4) P_{34}(x_3, x_4) \end{aligned}$$



System Diagram 7

$$\{125 : 25\}, \{345 : 35\}$$

$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{1|25}(x_1 \mid x_2, x_5) P_{4|35}(x_4 \mid x_3, x_5) P_{2|35}(x_2 \mid x_3, x_5) P_{35}(x_3, x_5) \end{aligned}$$

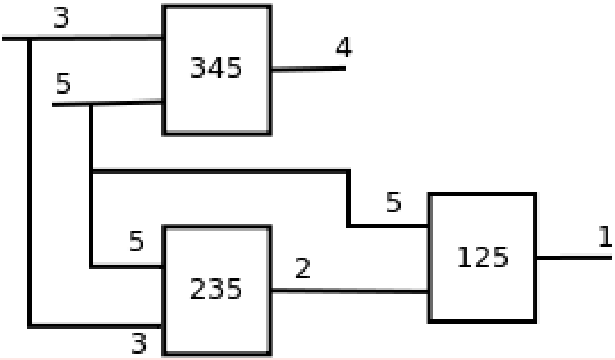


Figure: 3:System E

System Diagram 8

$$\{125 : 25\}, \{345 : 35\}$$

$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{1|25}(x_1 \mid x_2, x_5) P_{4|35}(x_4 \mid x_3, x_5) P_{3|25}(x_3 \mid x_2, x_5) P_{25}(x_2, x_5) \end{aligned}$$

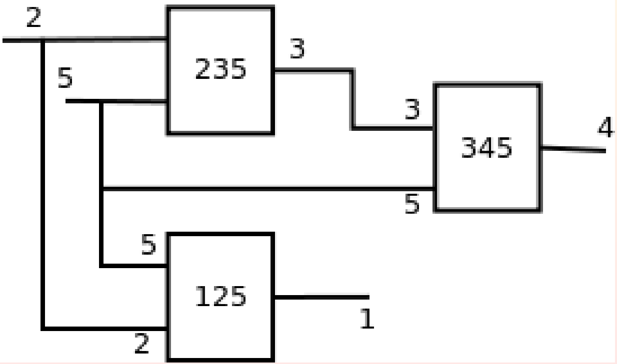


Figure: 3:System G

$$\{125 : 25\}, \{345 : 35\}$$

$$\begin{aligned} P_{12345}(x_1, x_2, x_3, x_4, x_5) &= \frac{P_{125}(x_1, x_2, x_5) P_{235}(x_2, x_3, x_5) P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5) P_{35}(x_3, x_5)} \\ &= P_{1|25}(x_1 \mid x_2, x_5) P_{4|35}(x_4 \mid x_3, x_5) P_{5|23}(x_5 \mid x_2, x_3) P_{23}(x_2, x_3) \end{aligned}$$

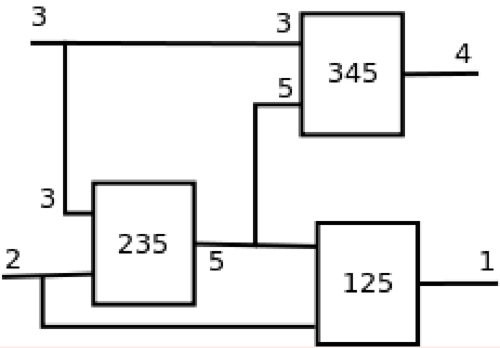


Figure: 3:System J

Feed Forward System Conditional Independences

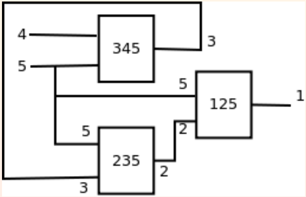
$$\begin{aligned}
P^{A}_{12345}(x_1,x_2,x_3,x_4,x_5) &= P_{45}(x_4,x_5)P_{3|45}(x_3|x_4,x_5)P_{1|25}(x_1|x_2,x_5)P_{2|35}(x_2|x_3,x_5)\\
P^{E}_{12345}(x_1,x_2,x_3,x_4,x_5) &= P_{35}(x_3,x_5)P_{4|35}(x_4|x_3,x_5)P_{1|25}(x_1|x_2,x_5)P_{2|35}(x_2|x_3,x_5)\\
P^{G}_{12345}(x_1,x_2,x_3,x_4,x_5) &= P_{25}(x_2,x_5)P_{3|25}(x_3|x_2,x_5)P_{1|25}(x_1|x_2,x_5)P_{4|35}(x_4|x_3,x_5)\\
P^{H}_{12345}(x_1,x_2,x_3,x_4,x_5) &= P_{15}(x_1,x_5)P_{2|15}(x_2|x_1,x_5)P_{3|25}(x_3|x_2,x_5)P_{4|35}(x_4|x_3,x_5)\\
P^{I}_{12345}(x_1,x_2,x_3,x_4,x_5) &= P_{12}(x_1,x_2)P_{5|12}(x_5|x_1,x_2)P_{3|25}(x_3|x_2,x_5)P_{4|35}(x_4|x_3,x_5)\\
P^{J}_{12345}(x_1,x_2,x_3,x_4,x_5) &= P_{23}(x_2,x_3)P_{1|25}(x_1|x_2,x_5)P_{5|23}(x_5|x_2,x_3)P_{4|35}(x_4|x_3,x_5)\\
P^{L}_{12345}(x_1,x_2,x_3,x_4,x_5) &= P_{34}(x_3,x_4)P_{1|25}(x_1|x_2,x_5)P_{2|35}(x_2|x_3,x_5)P_{5|34}(x_5|x_3,x_4)
\end{aligned}$$

These decompositions correspond to the same Decomposable Graphical Model

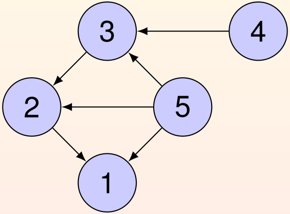
$$P_{12345}(x_1,x_2,x_4,x_4,x_5) = \frac{P_{345}(x_3,x_4,x_5)P_{125}(x_1,x_2,x_5)P_{235}(x_2,x_3,x_5)}{P_{25}(x_2,x_5)P_{35}(x_3,x_5)}$$

Feedforward Systems: Bayesian Networks

System A



Associated Bayesian Network



System A	$P(x_1, x_2, x_3, x_4, x_5)$	$=$	$P_{45}(x_4, x_5)P_{3\|45}(x_3 \mid x_4, x_5)P_{2\|35}(x_2 \mid x_3, x_5)P_{1\|25}(x_1 \mid x_2, x_5)$
Bayesian Network	$P(x_1, x_2, x_3, x_4, x_5)$	$=$	$P_4(x_4)P_5(x_5)P_{3\|45}(x_3 \mid x_4, x_5)P_{2\|35}(x_2 \mid x_3, x_5)P_{1\|25}(x_1 \mid x_2, x_5)$

System Structure and Decompositions

- $J = \{1, \ldots, N\}$
- Input set of subsystem k is I_k
- Output set of subsystem k is O_k
- $I_k \cup O_k = J_k$
- $I_k \cap O_k = \emptyset$
- $O_m \cap O_n = \emptyset$, $m \neq n$

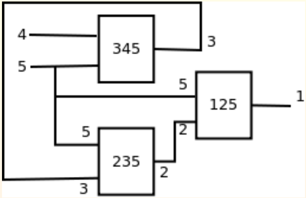
The system structure is defined by $\{(I_k, O_k, P_k)\}_{k=1}^{K}$

- Input Set I_k
- Output Set O_k
- Behavior P_k

$$P(x_j : j \in J) = P(x_m : m \in J - \cup_{k=1}^{K} O_k) \prod_{k=1}^{K} P_k(x_o : o \in O_k \mid x_i : i \in I_k)$$

The System Structure is Causal Structure

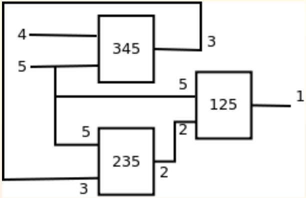
Causal Structure



System A:
4,5 are the direct cause of 3
2,5 are the direct cause of 1
3,5 are the direct cause of 2

$$
\begin{aligned}
J_1 &= \{3,4,5\} \\
I_1 &= \{4,5\} \\
O_1 &= \{3\} \\
J_2 &= \{1,2,5\} \\
I_2 &= \{2,5\} \\
O_2 &= \{1\} \\
J_3 &= \{2,3,5\} \\
I_3 &= \{3,5\} \\
O_3 &= \{2\}
\end{aligned}
$$

Causal Structure



System A:
4,5 are the direct cause of 3
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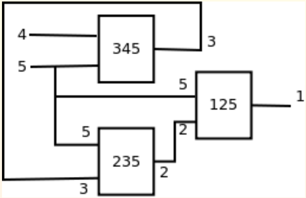
3,5 are the direct cause of 2

$$
\begin{aligned}
J_1 &= \{3,4,5\} \\
I_1 &= \{4,5\} \\
O_1 &= \{3\} \\
J_2 &= \{1,2,5\} \\
I_2 &= \{2,5\} \\
O_2 &= \{1\} \\
J_3 &= \{2,3,5\} \\
I_3 &= \{3,5\} \\
O_3 &= \{2\}
\end{aligned}
$$

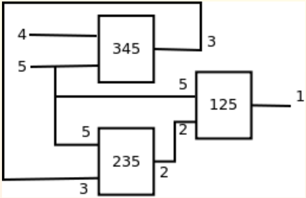
Causal Structure



System A

X_4, X_5 is the direct cause of X_3
X_2, X_5 is the direct cause of X_1
X_3, X_5 is the direct cause of X_2
X_4 is an indirect cause of X_1
X_1 has no causal influence on X_3: $X_1 \nrightarrow X_3$
X_3 has causal influence on X_1: $X_3 \rightarrow X_1$
Given X_2, X_5, X_3 has no causal influence on X_1: $X_3 \nrightarrow X_1 \mid X_2, X_5$
Given X_2, X_5, X_3 is conditionally independent of X_1: $X_3 \perp\!\!\!\perp X_1 \mid X_2, X_5$

Conditional Independence Structure



System A

X_4, X_5 is the direct cause of X_3
X_2, X_5 is the direct cause of X_1
X_3, X_5 is the direct cause of X_2
X_4 is an indirect cause of X_1
Given its parents, each variable is conditionally independent of its non-descendants
Given X_3 and X_5, X_2 is conditionally independent X_4: $X_2, \perp\!\!\!\perp X_4 \mid X_3, X_5$

Conditional Independence Structure

$$P_{12345}(x_1, x_2, x_4, x_4, x_5) = \frac{P_{345}(x_3, x_4, x_5)P_{125}(x_1, x_2, x_5)P_{235}(x_2, x_3, x_5)}{P_{25}(x_2, x_5)P_{35}(x_3, x_5)}$$

$$\begin{aligned}
P_{24|35}(x_2, x_4 \mid x_3, x_5) &= \sum_{x_1} \frac{P_{125}(x_1, x_2, x_5)P_{235}(x_2, x_3, x_5)P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5)P_{35}(x_3, x_5)P_{35}(x_3, x_5)} \\
&= \frac{P_{235}(x_2, x_3, x_5)P_{345}(x_3, x_4, x_5)}{P_{25}(x_2, x_5)P_{35}(x_3, x_5)P_{35}(x_3, x_5)} P_{25}(x_2, x_5) \\
&= \frac{P_{235}(x_2, x_3, x_5)P_{345}(x_3, x_4, x_5)}{P_{35}(x_3, x_5)P_{35}(x_3, x_5)} \\
&= P_{2|35}(x_2 \mid x_3, x_5)P_{4|35}(x_4 \mid x_3, x_5)
\end{aligned}$$

Digraphs, Feedforward, Feedback Systems

Let $\{(I_k, O_k, R_k)\}_{k=1}^{K}$ be a system.

- Input Set I_k
- Output Set O_k
- Behavior P_k

Define the associated system digraph (J, E) by

$$\begin{aligned} J &= \cup_{k=1}^{K} I_k \cup O_k \\ E &= \cup_{k=1}^{K} I_k \times O_k \end{aligned}$$

Definition

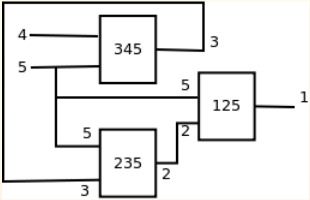
A system $\{(I_k, O_k, R_k)\}$ is called a feedforward system if and only if the digraph (J, E) is acyclic. A system that is not feedforward is called a feedback system.

Possible Causal System Structure

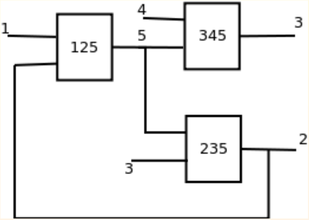
Let us consider all the possibilities where each subsystem has exactly one output variable and no two different subsystems produce the same output variables.

System	subsystem	output	subsystem	output	subsystem	output
A	345	3	235	2	125	1
B	345	3	235	2	125	5
C	345	3	235	5	125	1
D	345	3	235	5	125	2
E	345	4	235	2	125	1
F	345	4	235	2	125	5
G	345	4	235	3	125	1
H	345	4	235	3	125	2
I	345	4	235	3	125	5
J	345	4	235	5	125	1
K	345	4	235	5	125	2
L	345	5	235	2	125	1
M	345	5	235	3	125	1
N	345	5	235	3	125	2

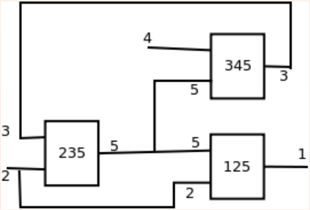
System Diagrams



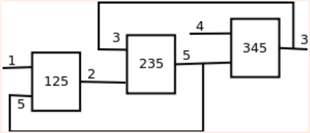
(a) System A: Feedfoward



(b) System B: Feedback

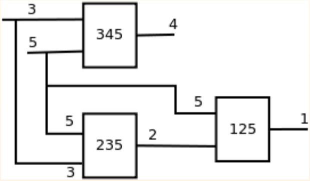


(c) System C: Feedback

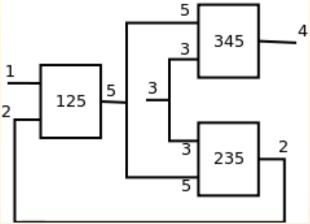


(d) System D: Feedback

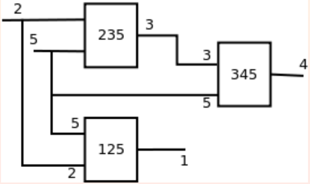
System Diagrams



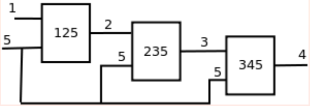
(e) System E: Feedfoward



(f) System F: Feedback

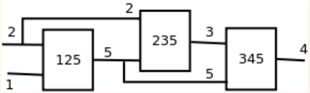


(g) System G: Feedfoward

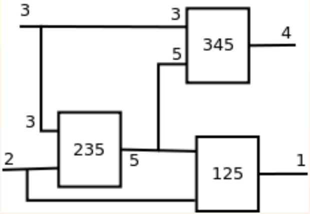


(h) System H: Feedfoward

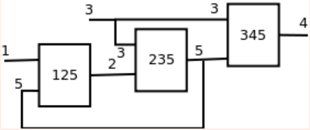
System Diagrams



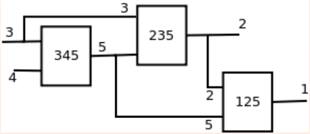
(i) System I: Feedfoward



(j) System J: Feedfoward

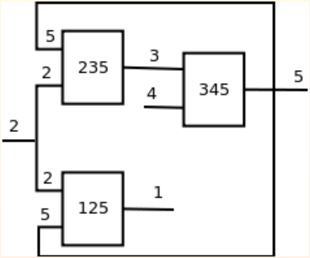


(k) System K: Feedback

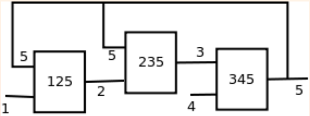


(l) System L: Feedfoward

System Diagrams



(m) System M: Feedback



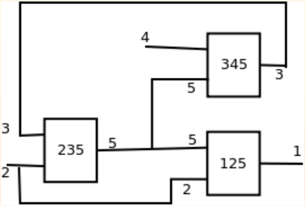
(n) System N: Feedfoward

Analysing Feedback Systems

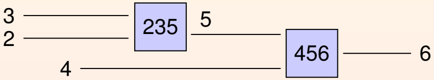
- Remove any subsystem not part of the feedback loop
- Break the feedback loop
  - This prevents the output variable y of the feedback loop to connect to a prior subsystem input variable x.
  - This makes the system a feedforward system
- Calculate the feedforward system behavior
- Add the equation $x = y$
- Calculate the new results

Feedback Systems

System C



System C with subsystem 125 removed and feedback loop broken output variable 3 renamed to 6

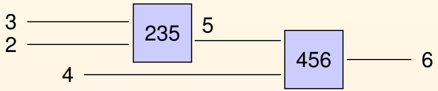


- Variable x_k has N_k possible values
- Fix variables $x_2 = a_2$ and $x_4 = a_4$
- Use a matrix notation

Matrix Notation Conventions

$P_6\Big\rvert^{N_6 \times 1}_{\substack{x_2 = a_2 \\ x_4 = a_4}}$	is the vector of probabilities for variable x_6 over its N_6 values with x_2 fixed at the value a_2 and x_4 fixed at the value a_4
$P_{5\|23}\Big\rvert^{N_5 \times N_3}_{x_2 = a_2}$	is the matrix of conditional probabilities of variable x_5 given x_3 with variable x_2 fixed at the value a_2
$P_{6\|45}\Big\rvert^{N_6 \times N_5}_{x_4 = a_4}$	is the matrix of conditional probabilities of variable x_6 given x_5 with variable x_4 fixed at the value a_4
$P_3\Big\rvert^{N_3 \times 1}_{\substack{x_2 = a_2 \\ x_4 = a_4}}$	is the vector of probabilities for variable x_3 over its N_3 values with x_2 fixed at the value a_2 and x_4 fixed at the value a_4

Reduced Feedforward System



The feedforward matrix equation relating the output variable x_6 to the input variable x_3 when input variable x_2 is fixed to value a_2 and input variable x_4 is fixed to value a_4 is then

$$P_6\Big|^{N_6\times 1}_{\substack{x_2=a_2\\ x_4=a_4}} = P_{6|45}\Big|^{N_6\times N_5}_{x_4=a_4} \; P_{5|23}\Big|^{N_5\times N_3}_{x_2=a_2} \; P_3\Big|^{N_3\times 1}_{\substack{x_2=a_2\\ x_4=a_4}}$$

Connecting The Feedback Loop

Set variable $x_6 = x_3$, noting that $N_6 = N_3$ and that variable x_6 and x_3 have the same range sets. The resulting matrix equation is

$$P_3\Big|^{N_3 \times 1}_{\substack{x_2 = a_2 \\ x_4 = a_4}} \quad = \quad P_{3|45}\Big|^{N_3 \times N_5}_{x_4 = a_4} \; P_{5|23}\Big|^{N_5 \times N_3}_{x_2 = a_2} \; P_3\Big|^{N_3 \times 1}_{\substack{x_2 = a_2 \\ x_4 = a_4}}$$

This equation can be easily solved for P_3 since it is the eigenvector corresponding to eigenvalue of 1 of the matrix

$$P_{3|45}\Big|^{N_3 \times N_5}_{x_4 = a_4} \; P_{5|23}\Big|^{N_5 \times N_3}_{x_2 = a_2}$$

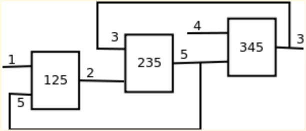
Computing Joint Probability

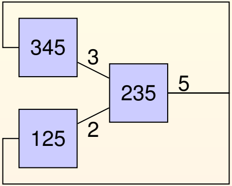
Thus for each different value of the externally set input variables x_2 and x_4, there will be different distribution for x_3. Once, the distribution of x_3 is known, the joint distribution of all variables, can be calculated by means of the corresponding conditional probabilities.

$P_3|^{N_3\times 1}_{\substack{x_2 = a_2 \\ x_4 = a_4}}$ is really the conditional probability $P_{3|24}(x_3|a_2, a_4)$.

$$P_{12345}(x_1, x_2, x_3, x_4, x_5) \quad = \quad P_{1|25}(x_1|x_2, x_5)P_{3|24}(x_3, |x_2, x_4)P_{5|23}(x_5|x_2, x_3)P_{24}(x_2, x_4)$$

Multiple Connected Feedback Loops



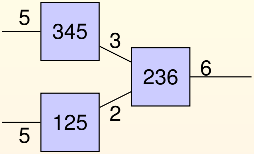


- Fix the external variables $x_1 = a_1$ and $x_4 = a_4$
- Take the combined variable (x_2, x_3) as the feedback variable
- The conditional probability matrix for (x_2, x_3) given x_5 is $N_2 N_3 \times N_5$.

$$P_{23|5}\Big|^{N_2 N_3 \times N_5}_{\substack{x_1 = a_1 \\ x_4 = a_4}} = P_{2|15}\Big|^{N_2 \times N_5}_{x_1 = a_1} \otimes P_{3|45}\Big|^{N_3 \times N_5}_{x_4 = a_4}$$

where $\otimes$ is the kronecker matrix product and simply allows us to denote a conditional probability matrix where one of the variables is the joint variable (x_2, x_3).

Multiple Connected Feedback Loops



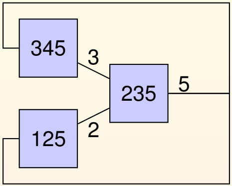
First we break the feedback loops and rename the output variable x_5 to x_6. Now we can write

$$P_6\Big|^{N_6\times 1}_{\substack{x_1=a_1\\x_4=a_4}} = P_{6|23}\Big|^{N_6\times N_2N_3}_{\substack{x_1=a_1\\x_4=a_4}} P_{23|5}\Big|^{N_2N_3\times N_5}_{\substack{x_1=a_1\\x_4=a_4}} P_5\Big|^{N_5\times 1}_{\substack{x_1=a_1\\x_4=a_4}}$$

Now we connect the feedback loop. We set variable $x_6 = x_5$, noting that $N_6 = N_5$ and that variable x_6 and x_5 have the same range sets. The resulting matrix equation is

$$P_5\Big|^{N_5\times 1}_{\substack{x_1=a_1\\x_4=a_4}} = P_{5|23}\Big|^{N_5\times N_2N_3}_{\substack{x_1=a_1\\x_4=a_4}} P_{23|5}\Big|^{N_2N_3\times N_5}_{\substack{x_1=a_1\\x_4=a_4}} P_5\Big|^{N_5\times 1}_{\substack{x_1=a_1\\x_4=a_4}}$$

Multiple Connected Feedback Loops

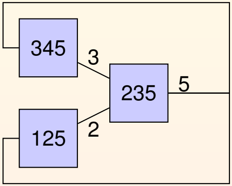


$$P_5\Big|^{N_5\times 1}_{\substack{x_1=a_1\\x_4=a_4}} = P_{5|23}\Big|^{N_5\times N_2N_3}_{\substack{x_1=a_1\\x_4=a_4}} P_{23|5}\Big|^{N_2N_3\times N_5}_{\substack{x_1=a_1\\x_4=a_4}} P_5\Big|^{N_5\times 1}_{\substack{x_1=a_1\\x_4=a_4}}$$

As before, this equation is easily solved as $P_5\Big|^{N_5\times 1}_{\substack{x_1=a_1\\x_4=a_4}}$ is just the eigenvector having eigenvalue 1 of the matrix

$$P_{5|23}\Big|^{N_5\times N_2N_3}_{\substack{x_1=a_1\\x_4=a_4}} P_{23|5}\Big|^{N_2N_3\times N_5}_{\substack{x_1=a_1\\x_4=a_4}}$$

Multiple Connected Feedback Loops: Joint Probability



$P_5\Big|^{N_5\times 1}_{\substack{x_1=a_1\\x_4=a_4}}$ is the conditional probability $P_{5|14}(x_5|a_1,a_4)$

$$P_{12345}(x_1,x_2,x_3,x_4,x_5) \quad = \quad P_{5|14}(x_5|x_1,x_4)P_{14}(x_1,x_4)P_{2|15}(x_2|x_1,x_5)P_{3|45}(x_3|x_4,x_5)$$