Parametric Probability Models

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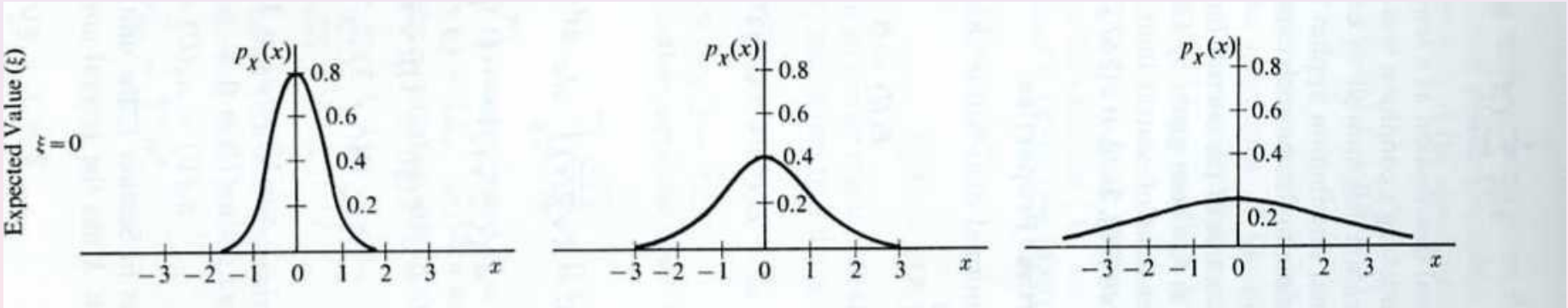
Outline

The Multivariate Normal Density Function

$$p(x) = \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu))$$



The Density Function

$$I = \int_x \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu))dx$$

Let $y = x - \mu$

$$I = \int_y \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-\frac{1}{2}y'\Sigma^{-1}y)dy$$

The Density Function

Now by the eigenvalue eigenvector decomposition of Σ, $\Sigma = T'DT$, where T is orthonormal and D is diagonal. Hence,

$$\begin{aligned}
\Sigma^{-1} &= (T'DT)^{-1} \\
&= T^{-1}D^{-1}(T')^{-1} \\
&= T'D^{-1}T \\
&= T'D'^{-\frac{1}{2}}D^{-\frac{1}{2}}T
\end{aligned}$$

so that

$$\begin{aligned}
I &= \int_y \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-y'T'D'^{-\frac{1}{2}}D^{-\frac{1}{2}}Tydy \\
&= \int_y \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-(D^{-\frac{1}{2}}Ty)'(D^{-\frac{1}{2}}Ty))dy
\end{aligned}$$

The Density Function

$$I = \int_y \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-(D^{-\frac{1}{2}}Ty)'(D^{-\frac{1}{2}}Ty))dy$$

Let $z = D^{-\frac{1}{2}}Ty$ so that $y = T'D^{\frac{1}{2}}z$. The Jacobian is $\frac{\partial y}{\partial z}$.

$$\begin{aligned}
\frac{\partial y}{\partial z} &= T'D^{\frac{1}{2}} \\
|\frac{\partial y}{\partial z}| &= |T'D^{\frac{1}{2}}| \\
&= |T'|\,|D^{\frac{1}{2}}| \\
&= |D|^{\frac{1}{2}}
\end{aligned}$$

The Density Function

$$I = \int_y \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-(D^{-\frac{1}{2}}Ty)'(D^{-\frac{1}{2}}Ty))dy$$

Hence,

$$I = \int_z \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-z'z)|D|^{\frac{1}{2}}dz$$

The Density Function

Recall,

$$\begin{aligned} \Sigma &= T'DT \\ |\Sigma| &= |T'DT| \\ &= |T'|\,|D|\,|T| \\ &= |D| \end{aligned}$$

Hence, $|\Sigma|^{\frac{1}{2}} = |D|^{\frac{1}{2}}$.

The Density Function

$$
\begin{aligned}
I &= \int_z \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-z'z)|D|^{\frac{1}{2}} dz \\
&= \int_z \frac{1}{|2\pi|^{\frac{N}{2}}} \exp(-z'z) dz \\
&= \prod_{n=1}^{N} \int_{z_n} \frac{1}{(2\pi)^{\frac{1}{2}}} \exp(-z_n^2) dz_n \\
&= \prod_{n=1}^{N} 1 \\
&= 1
\end{aligned}
$$

The Mean

$$
\begin{aligned}
p(x) &= \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)) \\
E[x] &= \int_x xp(x)dx \\
&= \int_x \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} x \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu))dx
\end{aligned}
$$

The Mean

Let $y = x - \mu$.
Note that odd functions integrated over even limits result in zero.

$$
\begin{aligned}
E[x] &= \int_y \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}}(y+\mu)\exp(-\frac{1}{2}y'\Sigma^{-1}y)dy \\
&= \int_y \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}}y\exp(-\frac{1}{2}y'\Sigma^{-1}y)dy + \\
&\qquad \int_y \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}}\mu\exp(-\frac{1}{2}y'\Sigma^{-1}y)dy \\
&= 0 + \mu\int_y \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}}\exp(-\frac{1}{2}y'\Sigma^{-1}y)dy \\
&= \mu
\end{aligned}
$$

The Covariance

$$
\begin{aligned}
Cov(x) &= E[(x-\mu)(x-\mu)'] \\
&= \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \int_x (x-\mu)(x-\mu)' \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu))dx \\
&= \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \int_y yy' \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-\frac{1}{2}y'\Sigma^{-1}y)dx
\end{aligned}
$$

The Covariance

Let $z = D^{-\frac{1}{2}} Ty$. Then $y = T' D^{\frac{1}{2}} z$ so that

$$
\begin{aligned}
Cov(x) &= \frac{1}{|2\pi|^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \int_z T' D^{\frac{1}{2}} z (T' D^{\frac{1}{2}} z)' \exp(-\frac{1}{2} z' z) |\Sigma|^{\frac{1}{2}} dz \\
&= \frac{1}{|2\pi|^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \int_z T' D^{\frac{1}{2}} z z' D^{\frac{1}{2}} T \exp(-\frac{1}{2} z' z) |\Sigma|^{\frac{1}{2}} dz \\
&= T' D^{\frac{1}{2}} \Big(\frac{1}{|2\pi|^{\frac{N}{2}} |\Sigma|^{\frac{1}{2}}} \int_z z z' \exp(-\frac{1}{2} z' z) |\Sigma|^{\frac{1}{2}} dz \Big) D^{\frac{1}{2}} T \\
&= \frac{1}{|2\pi|^{\frac{N}{2}}} T' D^{\frac{1}{2}} \Big(\int_z z z' \exp(-\frac{1}{2} z' z) dz \Big) D^{\frac{1}{2}} T
\end{aligned}
$$

The Covariance

$$
\begin{aligned}
Cov(x) &= \frac{1}{|2\pi|^{\frac{N}{2}}} T' D^{\frac{1}{2}} \Big(\int_z zz' \exp(-\frac{1}{2} z'z) dz \Big) D^{\frac{1}{2}} T \\
&= \frac{1}{|2\pi|^{\frac{N}{2}}} T' D^{\frac{1}{2}} (kI) D^{\frac{1}{2}} T \\
&= k \frac{1}{|2\pi|^{\frac{N}{2}}} T' D^{\frac{1}{2}} D^{\frac{1}{2}} T \\
&= k \frac{1}{|2\pi|^{\frac{N}{2}}} T' D T \\
&= k \frac{1}{|2\pi|^{\frac{N}{2}}} \Sigma \\
&= \Sigma
\end{aligned}
$$

Isodensity Contours

$$p(x) = \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu))$$

- μ mean vector
- Σ covariance matrix

The isodensity contours are defined by the sets $\{x \mid p(x) = constant\}$. If $p(x) = constant$, this implies that for some r,

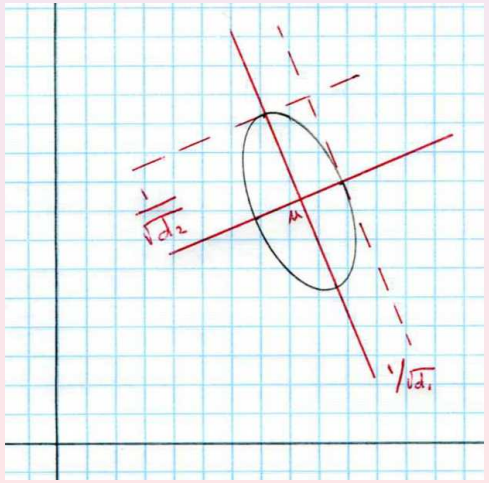
$$(x-\mu)'\Sigma^{-1}(x-\mu) = r^2$$

This is an equation of the surface of an ellipsoid with center μ.

The Ellipsoid

$$
\begin{aligned}
& \{x \mid (x-\mu)'\Sigma^{-1}(x-\mu) = r^2\} \\
= \; & \{x \mid (x-\mu)'T'D^{-1}T(x-\mu) = r^2\} \\
= \; & \{x \mid [D^{-\frac{1}{2}}T(x-\mu)]'[D^{-\frac{1}{2}}T(x-\mu)] = r^2\}
\end{aligned}
$$

If $d_1 > d_2$ and $D^{-1} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$



The Gaussian Classifier

The Gaussian Classifier assumes that the class conditional density functions are Multivariate Normal. It assigns a vector x to class c^1 when

$$\frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma_1|^{\frac{1}{2}}}\exp(-\frac{1}{2}(x-\mu_1)'\Sigma_1^{-1}(x-\mu_1))P(c^1) \quad >$$
$$\frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma_2|^{\frac{1}{2}}}\exp(-\frac{1}{2}(x-\mu_2)'\Sigma_2^{-1}(x-\mu_2))P(c^2)$$

The Gaussian Classifier

Assign vector x to class c_1 when

$$-\frac{1}{2}(x-\mu_1)'\Sigma_1^{-1}(x-\mu_1) - \frac{1}{2}\log|\Sigma_1| + \log P(c^1) \quad >$$
$$-\frac{1}{2}(x-\mu_2)'\Sigma_2^{-1}(x-\mu_2) - \frac{1}{2}\log|\Sigma_2| + \log P(c^2)$$

Assign vector x to class c_1 when

$$(x-\mu_1)'\Sigma_1^{-1}(x-\mu_1) + \log|\Sigma_1| - 2\log P(c^1) \quad <$$
$$(x-\mu_2)'\Sigma_2^{-1}(x-\mu_2) + \log|\Sigma_2| - 2\log P(c^2)$$

The Gaussian Classifier

When $|\Sigma_1| = |\Sigma_2|$ and $P(c^1) = P(c^2)$, then assign vector x to class c_1 when

$$(x - \mu_1)'\Sigma_1^{-1}(x - \mu_1) \quad < \quad (x - \mu_2)'\Sigma_2^{-1}(x - \mu_2)$$

The Gaussian Classifier

When $\Sigma_1 = \Sigma_2 = \Sigma$ and $P(c^1) = P(c^2)$, then assign vector x to class c_1 when

$$\begin{aligned}
(x-\mu_1)'\Sigma^{-1}(x-\mu_1) &< (x-\mu_2)'\Sigma^{-1}(x-\mu_2) \\
x'\Sigma^{-1}x - 2\mu_1'\Sigma^{-1}x + \mu_1'\Sigma^{-1}\mu_1 &< x'\Sigma^{-1}x - 2\mu_2'\Sigma^{-1}x + \mu_2'\Sigma^{-1}\mu_2 \\
-2\mu_1'\Sigma^{-1}x + \mu_1'\Sigma^{-1}\mu_1 &< -2\mu_2'\Sigma^{-1}x + \mu_2'\Sigma^{-1}\mu_2 \\
2(\mu_2-\mu_1)'\Sigma^{-1}x &< \mu_2'\Sigma^{-1}\mu_2 - \mu_1'\Sigma^{-1}\mu_1 \\
(\mu_2-\mu_1)'\Sigma^{-1}x &< (\mu_2-\mu_1)'\Sigma^{-1}(\frac{\mu_1+\mu_2}{2}) \\
(\mu_2-\mu_1)'\Sigma^{-1}(x-\frac{\mu_1+\mu_2}{2}) &< 0
\end{aligned}$$

Fisher Linear Discriminant

$$v = \Sigma_W^{-1}(\mu_1 - \mu_2)$$

Assign x to class 1 if

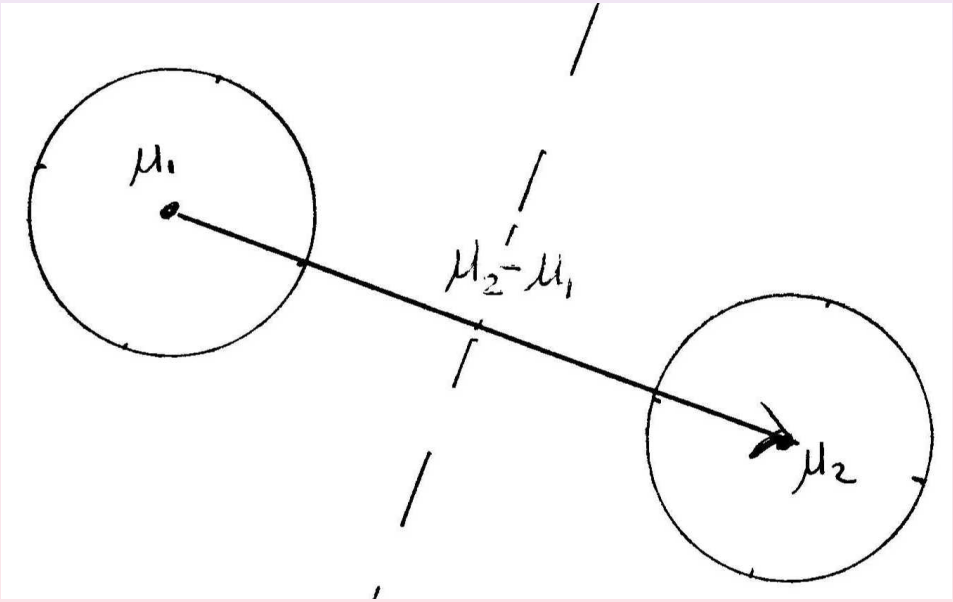
$$\begin{aligned} v'x &\geq \theta \\ \left(\Sigma_W^{-1}(\mu_1 - \mu_2)\right)' x &\geq \theta \\ (\mu_1 - \mu_2)'\Sigma_W^{-1\prime} x &\geq \theta \\ (\mu_1 - \mu_2)'\Sigma_W^{-1} x &\geq \theta \\ (\mu_2 - \mu_1)'\Sigma_W^{-1} x &< \theta \end{aligned}$$

When $\Sigma_1 = \Sigma_2$, the Gaussian classifier is a linear classifier and identical to the Fisher Linear Discriminant Classifier since $\Sigma_W = \Sigma_1 = \Sigma_2$

The Gaussian Classifier

When $\Sigma = I$, assign vector x to class c_1 when

$$(\mu_2 - \mu_1)'(x - \frac{\mu_1 + \mu_2}{2}) \quad < \quad 0$$



Pearson Distribution System

$$\begin{aligned}
\frac{d}{dx}\log p(x) &= -\frac{x-a}{b_2x^2+b_1x+b_0} \\
\log p(x) &= -\int \frac{x-a}{b_2x^2+b_1x+b_0}dx + A \\
p(x) &= Ke^{-\int \frac{x-a}{b_2x^2+b_1x+b_0}dx}
\end{aligned}$$

Assuming $b_2 \neq 0$, use partial fraction expansion

$$p(x) = Ke^{-\int \frac{1}{b(a_1-a_2)}\left[\frac{a+a_1}{x-a_1}-\frac{a+a_2}{x-a_2}\right]dx}$$

Pearson Type I

$a_1 < 0 < a_2$

$$
\begin{aligned}
p(x) &= K(x-a_1)^{m_1}(a_2-x)^{m_2},\ a_1 \le x \le a_2 \\
m_1 &= \frac{a+a_1}{b(a_2-a_1)} \\
m_2 &= -\frac{a+a_2}{b(a_2-a_1)}
\end{aligned}
$$

Translating and scaling to put $0 \le x \le 1$

$$
p(x) = \frac{1}{B(m,n)} x^{m-1}(1-x)^{n-1}
$$

Pearson Type II

$a_1 < 0 < a_2$ and $m = m_1 = m_2$

$$p(x) = K(x - a_1)^m (a_2 - x)^m, \; a_1 \leq x \leq a_2$$

Translating and scaling to put $0 \leq x \leq 1$

$$p(x) = \frac{1}{B(m, m)} x^{m-1} (1 - x)^{m-1}$$

Pearson Type III

$b_2 = 0$ and $b_1 \neq 0$

$$
\begin{aligned}
\frac{d}{dx}\log p(x) &= -\frac{x-a}{b_1 x + b_0} \\
&= -\left[\frac{1}{b_1} - \frac{a + b_0/b_1}{b_0 + b_1 x}\right] \\
\log p(x) &= -\int \frac{1}{b_1} - \frac{a + b_0/b_1}{b_0 + b_1 x} dx + A \\
p(x) &= K e^{-\int \frac{1}{b_1} - \frac{a+b_0/b_1}{b_0+b_1 x} dx} \\
&= K(b_0 + b_1 x)^m e^{-x/b_1},\ x \geq -\frac{b_0}{b_1},\ \text{when } b_1 > 0 \\
m &= b_1^{-1}(b_0 b_1^{-1} + a)
\end{aligned}
$$

Pearson Type III

Translating and scaling so that $x \geq 0$

$$p(x) = \frac{1}{\Gamma(\lambda)} x^{\lambda-1} e^{-x}, \ \lambda > 0, \ 0 \leq x$$

Pearson Type IV

Roots are complex

$$\begin{aligned}\frac{d}{dx}\log p(x) &= -\frac{x-a}{b_2x^2+b_1x+b_0}\\ &= -\frac{x-a}{c_0+b_2(x+c_1)^2}\end{aligned}$$

$$p(x) = K[c_0+b_2(x+c_1)^2]^{-(2b_2)^{-1}}e^{\frac{a+c_1}{\sqrt{b_2c_0}}\tan^{-1}\frac{x+c_1}{\sqrt{c_0/b_2}}}$$

Translating and scaling,

$$p(x) = K\left(1+\frac{x^2}{\sigma^2}\right)^{-m}e^{-\nu\tan^{-1}\frac{x}{\sigma}}$$

Pearson Type V

$b_2x^2 + b_1x + b_0$ is a perfect square

$$\begin{aligned} \frac{d}{dx}\log p(x) &= -\frac{x-a}{b_2(x+c)^2} \\ p(x) &= K(x+c)^{-\frac{1}{b_2}} e^{\frac{a+c}{b_2(x+c)}},\ x \geq -c \end{aligned}$$

Pearson Type VI

Roots of $b_2x^2 + b_1x + b_0$ are real and negative; $a_1 < a_2 < 0$

$$\begin{aligned} p(x) &= K(x-a_1)^{m_1}(x-a_2)^{m_2},\ x > a_2 \\ m_1 &= < -1 \\ m_1 + m_2 &< 0 \end{aligned}$$

Pearson Type VII

$b_1 = a = 0,\ b_0, b_2 > 0$

$$\begin{aligned} \frac{d}{dx} \log p(x) &= -\frac{x}{b_2 x^2 + b_0} \\ p(x) &= K(b_0 + b_2 x^2)^{-\frac{1}{2b_2}} \end{aligned}$$

The Metric

Definition

A function d is called a metric on a set X if and only if for every $x, y, z \in X$

- $d(x, y) \geq 0$
- $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

The space (X, d) is called a metric space.

Norm

Definition

The *norm* of a vector x is defined by

$$||x|| = \sqrt{x'x}$$

The Schwarz Inequality

Theorem

$$|a'b| \leq ||a||\ ||b||$$

Proof:

$$\sum_{i=1}^{N} a_i^2 \sum_{j=1}^{N} b_j^2 - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (a_i b_j - b_i a_j)^2$$

$$= \sum_{i=1}^{N} a_i^2 \sum_{j=1}^{N} b_j^2 - \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (a_i^2 b_j^2 - 2 a_i b_j b_i a_j + b_i^2 a_j^2)$$

Schwarz Inequality

$$
\begin{aligned}
&= \sum_{i=1}^{N} a_i^2 \sum_{j=1}^{N} b_j^2 - \frac{1}{2} \sum_{i=1}^{N} (a_i^2 \sum_{j=1}^{N} b_j^2 - 2a_i b_i \sum_{j=1}^{N} a_j b_j + b_i^2 \sum_{j=1}^{N} a_j^2) \\
&= \sum_{i=1}^{N} a_i^2 \sum_{j=1}^{N} b_j^2 - \frac{1}{2} \left(\sum_{i=1}^{N} a_i^2 \sum_{j=1}^{N} b_j^2 - 2 \sum_{i=1}^{N} a_i b_i \sum_{j=1}^{N} a_j b_j + \right. \\
&\qquad \left. \sum_{i=1}^{N} b_i^2 \sum_{j=1}^{N} a_j^2 \right) \\
&= \left(\sum_{i=1}^{N} a_i b_i \right)^2
\end{aligned}
$$

Schwarz Inequality

$$
\begin{aligned}
\left(\sum_{i=1}^{N} a_i b_i\right)^2 &= \sum_{i=1}^{N} a_i^2 \sum_{j=1}^{N} b_j^2 - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}(a_i b_j - b_i a_j)^2 \\
&\leq \sum_{i=1}^{N} a_i^2 \sum_{j=1}^{N} b_j^2 \\
(a'b)^2 &\leq ||a||^2 \, ||b||^2
\end{aligned}
$$

Norm

Theorem

$$||a+b|| \leq ||a|| + ||b||$$

Proof:

$$\begin{aligned}
||a+b||^2 &= (a+b)'(a+b) = a'a + 2a'b + b'b \\
&= ||a||^2 + 2a'b + ||b||^2 \\
&\leq ||a||^2 + 2|a'b| + ||b||^2 \\
&\leq ||a||^2 + 2||a||\,||b|| + ||b||^2 \\
&\leq (||a|| + ||b||)^2
\end{aligned}$$

Norms and Metrics

Theorem

The norm of a vector difference is a metric. Define $d(x, y) = ||x - y||$. *Then* d *is a metric.*

$$
\begin{aligned}
d(x, y) &= ||x - y|| = \sqrt{(x - y)'(x - y)} \geq 0 \\
0 &= ||x - y|| \text{ if and only if } x - y = 0 \\
d(x, y) &= ||x - y|| = \sqrt{(x - y)'(x - y)} \\
&= \sqrt{(y - x)'(y - x)} = d(y, x) \\
d(x, y) &= ||x - y|| = ||(x - z) + (z - y)|| \\
&\leq ||x - z|| + ||z - y|| = d(x, z) + d(z, y)
\end{aligned}
$$

Quadratic Forms and Metrics

Theorem

Let A be a positive definite symmetric matrix. Define $\rho(x,y) = \sqrt{(x-y)'A(x-y)}$, then ρ is a metric.

Proof: Let $A = T'DT$. Then

$$\begin{aligned}
\rho^2(x,y) &= (x-y)'A(x-y) = (x-y)'T'DT(x-y) \\
&= (x-y)'T'D^{\frac{1}{2}}D^{\frac{1}{2}}T(x-y) = \\
&= (D^{\frac{1}{2}}T(x-y))'(D^{\frac{1}{2}}T(x-y)) \\
&= \left((D^{\frac{1}{2}}Tx) - (D^{\frac{1}{2}}Ty)\right)' \left((D^{\frac{1}{2}}Tx) - (D^{\frac{1}{2}}Ty)\right) \\
&= d^2(D^{\frac{1}{2}}Tx, D^{\frac{1}{2}}Ty)
\end{aligned}$$

Quadratic Forms and Metrics

$$\rho(x, y) = d(D^{\frac{1}{2}}Tx, D^{\frac{1}{2}}Ty)$$

(1)
Since $d(u, v) \geq 0$ for every u, v,
$d(D^{\frac{1}{2}}Tx, D^{\frac{1}{2}}Ty) \geq 0$ so that $\rho(x, y) \geq 0$.

$\rho(x, y) = 0$ if and only if $d(D^{\frac{1}{2}}Tx, D^{\frac{1}{2}}Ty) = 0$.
And $d(D^{\frac{1}{2}}Tx, D^{\frac{1}{2}}Ty) = 0$ if and only if $D^{\frac{1}{2}}Tx = D^{\frac{1}{2}}Ty$.
$D^{\frac{1}{2}}Tx = D^{\frac{1}{2}}Ty$ if and only if $x = y$.

Quadratic Forms and Metrics

$$\rho(x, y) = d(D^{\frac{1}{2}} Tx, D^{\frac{1}{2}} Ty) \tag{2}$$

$$\begin{aligned} \rho(x, y) &= d(D^{\frac{1}{2}} Tx, D^{\frac{1}{2}} Ty) \\ &= d(D^{\frac{1}{2}} Ty, D^{\frac{1}{2}} Tx) \\ &= \rho(y, x) \end{aligned}$$

Quadratic Forms and Metrics

$$\rho(x, y) = d(D^{\frac{1}{2}} Tx, D^{\frac{1}{2}} Ty)$$

(3)

$$\begin{aligned} \rho(x, y) &= d(D^{\frac{1}{2}} Tx, D^{\frac{1}{2}} Ty) \\ &\leq d(D^{\frac{1}{2}} Tx, D^{\frac{1}{2}} Tz) + d(D^{\frac{1}{2}} Tz, D^{\frac{1}{2}} Ty) \\ &\leq \rho(x, z) + \rho(z, y) \end{aligned}$$

Multivariate Normal

$$
\begin{aligned}
p(x) &= \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)) \\
&= \frac{1}{|2\pi|^{\frac{N}{2}}|\Sigma|^{\frac{1}{2}}} \exp(-\frac{1}{2}\rho(x,\mu;\Sigma^{-1})^2
\end{aligned}
$$

Notice that $\exp(-\frac{1}{2}u^2)$ is a monotonically decreasing function of u. The Multivariate Normal density is just one density that converts a distance to a density through a monotonically decreasing function.

General Ellipsoidally Symmetric Multivariate Forms

Let f be any non-negative decreasing function in its tail and A a symmetric positive definite matrix. Then

$$p(x) = Kf\left(\sqrt{(x-\mu)'A(x-\mu)}\right)$$

with an appropriate value for K is a multivariate density function.

General Ellipsoidally Symmetric Multivariate Forms

$f : [0, \infty] \to [0, \infty]$ monotonically decreasing in its tail with

$$\int_r r^{N+1} f(r) dr \quad < \quad \infty$$

Define

$$p(x) \quad = \quad Kf\left(\sqrt{(x-\mu)' A (x-\mu)}\right)$$

If $\int p(x) dx = 1$, then

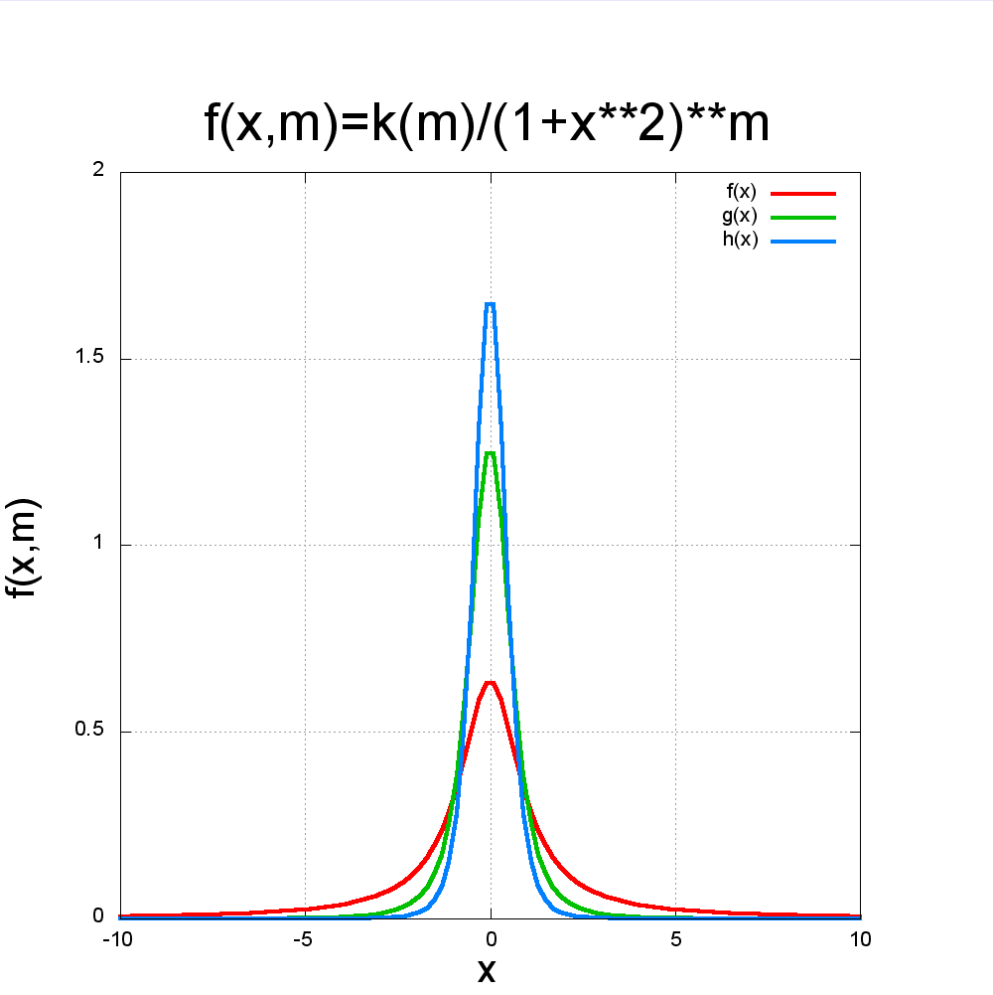
$$K \quad = \quad \frac{\Gamma(N/2)}{2\pi^{N/2} \int_r r^{N-1} f(r) dr} |A|^{\frac{1}{2}}$$

$$A \quad = \quad \frac{\int_r r^{N+1} f(r) dr}{N \int_r r^{N-1} f(r) dr} \Sigma^{-1}$$

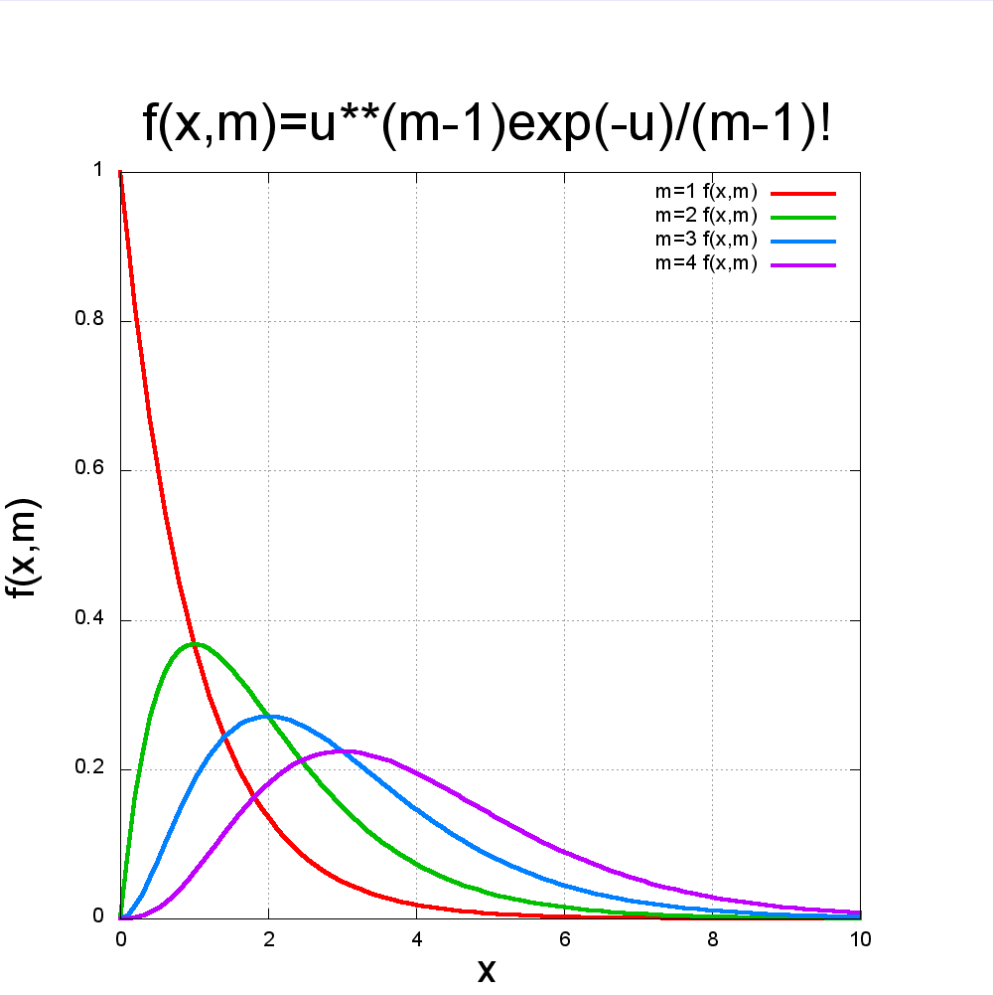
Ellipsoidally Symmetric Multivariate Forms

f	K	A
e^{-u^2}	$\frac{1}{(2\pi)^{N/2}}\lvert A\rvert^{\frac{1}{2}}$	Σ^{-1}
$(1+u^2)^{-m}$ $2m-N-2>0$	$\frac{\Gamma(m)}{\pi^{N/2}\Gamma(m-N/2)}\lvert A\rvert^{\frac{1}{2}}$	$\frac{1}{2m-N-2}\Sigma^{-1}$
$u^{m-1}e^{-u}$ $N+m-1>0$	$\frac{\Gamma(N/2)}{2\pi^{N/2}\Gamma(N+m-1)}\lvert A\rvert^{\frac{1}{2}}$	$\frac{(N+m-1)(N+m-2)}{N}\Sigma^{-1}$

Functional Forms



Gamma Distribution



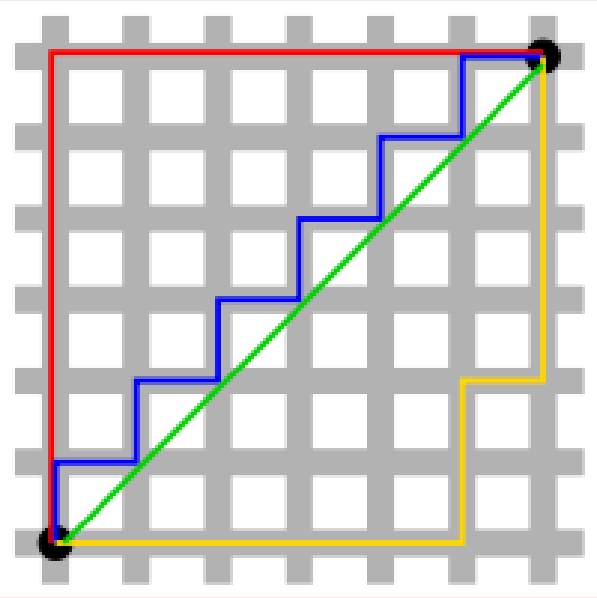
Metrics

$$
\begin{aligned}
\rho_{Manhattan}(x, y) &= \sum_{n=1}^{N} |x_n - y_n| \\
\rho_{Euclidean}(x, y) &= ||x - y||^2 \\
\rho_{Chebyshev}(x, y) &= \max_{n=1,\ldots,N} |x_n - y_n|
\end{aligned}
$$

Manhattan Metric

$$\rho_{Manhattan}(x, y) = \sum_{n=1}^{N} |x_n - y_n|$$

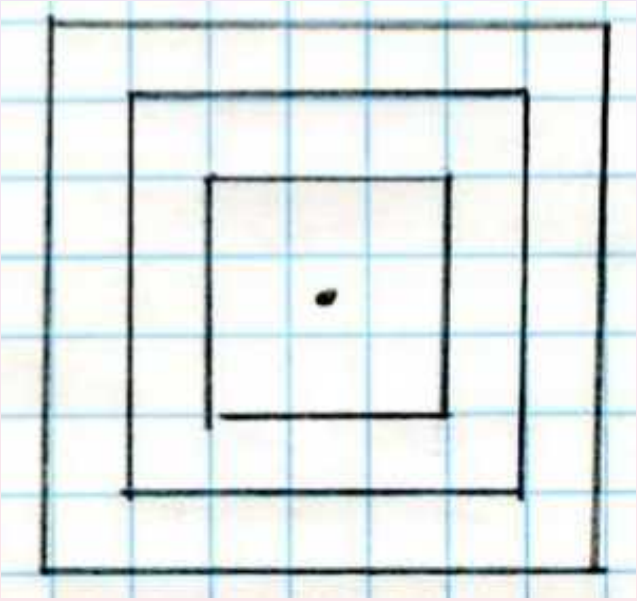
Also called the taxi cab metric and the city block metric.



Chebyshev

$$\rho_{Chebyshev}(x, y) = \max_{n=1,\ldots,N} |x_n - y_n|$$

Also called the chess board distance.



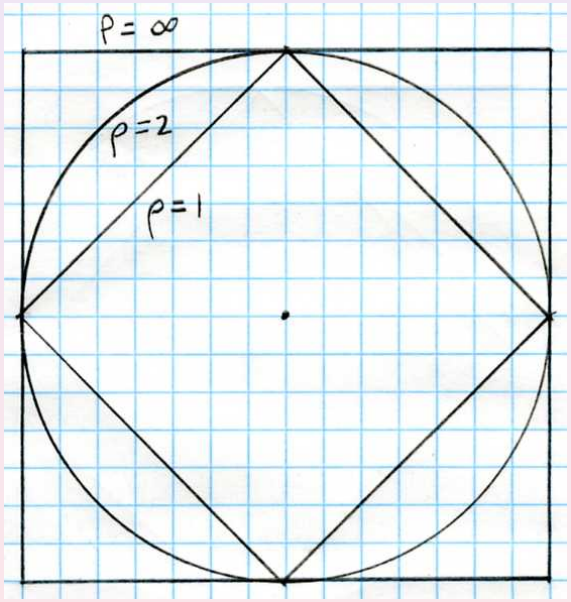
Minkowski Distance

$$\rho_{Minkowski}(x, y) = \left(\sum_{n=1}^{N} |x_n - y_n|^p \right)^{\frac{1}{p}}$$

- $p = 1$ Manhattan
- $p = 2$ Euclidean
- $p \to \infty$ Chebyshev

Minkowski Distance

$$\rho_{Minkowski}(x, y) = \left(\sum_{n=1}^{N} |x_n - y_n|^p \right)^{\frac{1}{p}}$$



Minkowski Distance

$$\rho_{Minkowski}(x, y) = \left(\sum_{n=1}^{N} |x_n - y_n|^p \right)^{\frac{1}{p}}$$

Let $|x_k - y_k| > |x_n - y_n|, \; n = 1, \ldots, N$

$$\begin{aligned} \lim_{p \to \infty} \left(\sum_{n=1}^{N} |x_n - y_n|^p \right)^{\frac{1}{p}} &= |x_k - y_k| \lim_{p \to \infty} \left[\sum_{n=1}^{N} \left(\frac{|x_n - y_n|}{|x_k - y_k|} \right)^p \right]^{\frac{1}{p}} \\ &= |x_k - y_x| \end{aligned}$$

General Parametric Probability Densities

Let f_k be a strictly monotonically decreasing function whose (N+2) order moment exists. Define

$$p(x \mid c^k) = m_k f_k(\rho_k(x, \mu_k))$$

Assign a vector x to class c_1 when

$$m_1 f_1(\rho_1(x, \mu_1))P(c^1) > m_2 f_2(\rho_2(x, \mu_2))P(c^2)$$

Using Only Distance Functions

Consider

$$\begin{pmatrix} \rho_1(x, \mu_1) \\ \rho_2(x, \mu_2) \end{pmatrix}$$

to be a feature vector.

Training Data: 2D Features

$$< x_1, \ldots, x_M > \text{class 1}$$
$$< y_1, \ldots, y_N > \text{class 2}$$

Form the class 1 feature vectors

$$\begin{pmatrix} \rho_1(x_1, \mu_1) \\ \rho_2(x_1, \mu_2) \end{pmatrix}, \ldots, \begin{pmatrix} \rho_1(x_M, \mu_1) \\ \rho_2(x_M, \mu_2) \end{pmatrix}$$

Form the class 2 feature vectors

$$\begin{pmatrix} \rho_1(y_1, \mu_1) \\ \rho_2(y_1, \mu_2) \end{pmatrix}, \ldots, \begin{pmatrix} \rho_1(y_N, \mu_1) \\ \rho_2(y_N, \mu_2) \end{pmatrix}$$

Quantize

Let q_i be a quantizer for feature i.

Class 1 feature vectors

$$\begin{pmatrix} q_1(\rho_1(x_1, \mu_1)) \\ q_2(\rho_2(x_1, \mu_2)) \end{pmatrix}, \ldots, \begin{pmatrix} q_1(\rho_1(x_M, \mu_1)) \\ q_2(\rho_2(x_M, \mu_2)) \end{pmatrix}$$

Class 2 feature vectors

$$\begin{pmatrix} q_1(\rho_1(y_1, \mu_1)) \\ q_2(\rho_2(y_1, \mu_2)) \end{pmatrix}, \ldots, \begin{pmatrix} q_1(\rho_1(y_N, \mu_1)) \\ q_2(\rho_2(y_N, \mu_2)) \end{pmatrix}$$

Use discrete probability Bayesian classification methodology

Training Data: 1D Features

$$< x_1, \ldots, x_M > \text{class 1}$$
$$< y_1, \ldots, y_N > \text{class 2}$$

Form the class 1 feature values

$$(\rho_1(x_1, \mu_1)), \ldots, (\rho_1(x_M, \mu_1))$$

Form the class 2 feature values

$$(\rho_2(y_1, \mu_2)), \ldots, (\rho_2(y_N, \mu_2))$$

Quantize

Same quantizer q for both classes

Form the class 1 feature values

$$(q(\rho_1(x_1, \mu_1))), \ldots, (q(\rho_1(x_M, \mu_1)))$$

Form the class 2 feature values

$$(q(\rho_2(y_1, \mu_2))), \ldots, (q(\rho_2(y_N, \mu_2)))$$

Non-parametric Probability Model

- Quantize the feature values for class 1 and class 2
- Construct the histogram h_1 for the class 1 feature values
- Construct the histogram h_2 for the class 2 feature values
- Normalize histograms so that they each sum to 1
- $P(c^1, x) = h_1(q(\rho(x, \mu_1)))P(c^1)$
  $P(c^2, x) = h_2(q(\rho(x, \mu_2)))P(c^2)$
- Use discrete probability Bayesian classification